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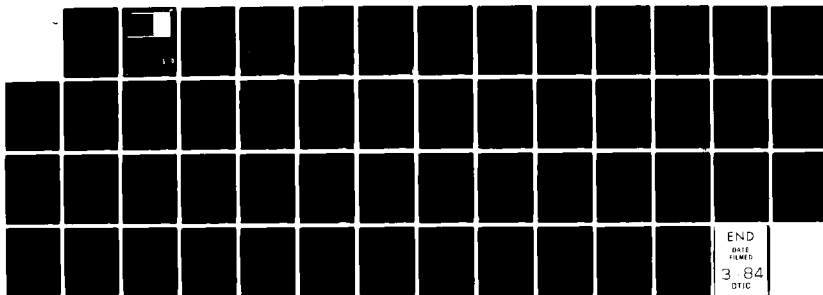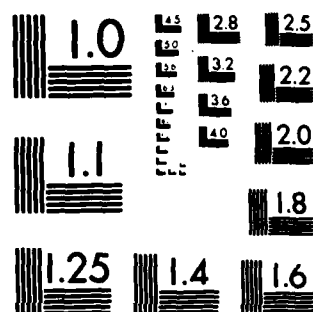   MATHEMATICS RESEARCH CENTER D L RUSSELL DEC 83
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THE DIRICHLET-NEUMANN BOUNDARY CONTROL
PROBLEM ASSOCIATED WITH MAXWELL'S
EQUATIONS IN A CYLINDRICAL REGION

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ABSTRACT

In a cylindrical region we consider, the author, electromagnetic fields independent of
the axial coordinate: controlling the time evolution of such fields by means
of boundary currents, likewise independent of the axial direction, is
equivalent to controlling, simultaneously, two wave equations; one with
boundary control of Dirichlet type, the other of Neumann type. In this paper
we provide a preliminary study of control problems of this type and indicate
what is necessary for extensions of our work.

SIGNIFICANCE AND EXPLANATION

This paper concerns the controllability of the Maxwell electromagnetic equations in a cylindrical spatial region by means of controlling currents caused to flow on the boundary of the region. Here controllability refers to the ability to transfer from electric and magnetic fields, given at the initial instant, to corresponding fields prescribed at a later instant.

Studies of this type are significant in relation to wave guides, EM-pulse devices, radar non-relective (stealth) aircraft, controlled thermonuclear fusion and many other important applications.

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THE DIRICHLET-NEUMANN BOUNDARY CONTROL PROBLEM ASSOCIATED WITH MAXWELL'S EQUATIONS IN A CYLINDRICAL REGION

D. L. Russell

1. BACKGROUND.

In this paper we consider a region $\Omega \subseteq R^3$, not necessarily bounded, having piecewise smooth boundary Γ and almost everywhere uniquely defined unit exterior normal vector $\vec{\nu} = \vec{\nu}(x,y,z)$, $(x,y,z) \in \Gamma$. It is assumed that the region Ω is occupied by a medium having constant electrical permitivity ε and constant magnetic permeability μ. We have then, in Ω, the paired electric and magnetic fields

$$\vec{E} = \vec{E}(x,y,z,t) ,$$

$$\vec{H} = \vec{H}(x,y,z,t) ,$$

having finite _energy_

$$E(t) = \tfrac{1}{2} \iiint\limits_{\Omega} (\varepsilon |\vec{E}|^2 + \mu |\vec{H}|^2) dv , \tag{1.1}$$

where $|\ |$ denotes the usual Euclidean norm in R^3. As is well known ([4], [9]), $\vec{E}$ and $\vec{H}$ satisfy, in Ω, Maxwell's equations

$$\operatorname{curl} \vec{H} = e \frac{\partial \vec{E}}{\partial t} , \tag{1.2}$$

$$\operatorname{curl} \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} , \tag{1.3}$$

$$\operatorname{div} \vec{E} = \rho , \tag{1.4}$$

$$\operatorname{div} \vec{H} = 0 , \tag{1.5}$$

where $\rho = \rho(x,y,z,t)$ is the electrical charge density in Ω - which is zero throughout this paper. (That equation (1.5) might eventually have to be modified to account for magnetic monopoles will trouble us not at all here!)

Control problems associated with Maxwell's equations have been of interest primarily in connection with nuclear fusion applications - in which case ρ is not identically equal to zero and the Maxwell equations are coupled with the dynamical equations governing the plasma evolution. In this connection we cite the work of P. K. C. Wang [29], [30], [31]. The point of view which we take here is that we cannot hope to treat these more complicated problems until we have a firmer grasp on the control theory of Maxwell's system in its own right. In this direction some work on controllability with control influence distributed throughout Ω has been carried out by G. Chen [2], [3]. We are primarily concerned here with the possibility of influencing the evolution of the fields $\vec{E}$ and $\vec{H}$ by means of an externally determined current $\vec{J}(x,y,z,t)$ flowing tangentially in Γ so that

$$\vec{J}(x,y,z,t)\cdot\vec{v}(x,y,z) = 0 , \tag{1.6}$$

for $(x,y,z) \in \Gamma$ where $\vec{v}(x,y,z)$ is defined. We will assume that the normal component of $\vec{E}$ vanishes outside Ω and that no charge is permitted to accumulate on Γ. Then we have the boundary conditions (see e.g. [4], [28])

$$\varepsilon\vec{E}(x,y,z,t)\cdot\vec{v}(x,y,z) = 0 \tag{1.7}$$

$$\mu\vec{H}_\tau(x,y,z,t) = \vec{v}(x,y,z) \times \vec{J}(x,y,z,t) \tag{1.8}$$

for $(x,y,z) \in \Gamma$ such that $\vec{v}(x,y,z)$ is well-defined. Here, and subsequently, the subscript τ refers to the component of the vector in question which is tangential to Γ. Similarly, the subscript ν will denote the normal component (thus (1.7) is the same as $\vec{E}_\nu = 0$). Writing

$$\vec{E} = \vec{E}_\nu + \vec{E}_\tau = \vec{E}_\tau \quad \text{on} \quad \Gamma ,$$
$$\vec{H} = \vec{H}_\nu + \vec{H}_\tau$$
$$\vec{J} = \vec{J}_\nu + \vec{J}_\tau = \vec{J}_\tau \quad \text{on} \quad \Gamma ,$$

we see that (1.8) becomes $\mu\vec{H}_\tau = \vec{v} \times \vec{J}_\tau$, so that $\vec{H}_\tau$ is a vector tangential to Γ and perpendicular to $\vec{J} = \vec{J}_\tau$.

The state space in which we study solutions of the above system will be denoted by $H_{E,d}(\Omega)$; it is a closed subspace of the space $H_E(\Omega)$ of square integrable six-dimensional fields $(\vec{E}(x,y,z,t), \vec{H}(x,y,z,t))$ with the inner product and norm

$$\langle (\vec{E}_1, \vec{H}_1); (\vec{E}_2, \vec{H}_2) \rangle \equiv \iiint\limits_{\Omega} (\epsilon \vec{E}_1 \cdot \vec{E}_2 + \mu \vec{H}_1 \cdot \vec{H}_2) dv$$

$$|(\vec{E}, \vec{H})|^2 = \langle (\vec{E}, \vec{H}); (\vec{E}, \vec{H}) \rangle . \tag{1.9}$$

Clearly $H_E(\Omega)$ is a real Hilbert space with this inner product. Where a complex space is required, we employ conjugation as usual. The state space $H_{E,d}(\Omega)$ is the closed span in $H_E(\Omega)$ of those continuously differentiable fields $(\vec{E}(x,y,z,t), \vec{H}(x,y,z,t))$ for which

$$\text{div } \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 ,$$

$$\text{div } \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 .$$

If $\vec{E}_0, \vec{H}_0$ and $\vec{E}_1, \vec{H}_1$ are two smooth solution pairs for (1.2)-(1.5), (1.7), (1.8), the first corresponding to $\vec{J} \equiv 0$ on Γ, we see easily that

$$\frac{d}{dt} \langle (\vec{E}_0, \vec{H}_0); (\vec{E}_1, \vec{H}_1) \rangle =$$

$$= \iiint\limits_{\Omega} (\epsilon [\vec{E}_0 \cdot \frac{\partial \vec{E}_1}{\partial t} + \frac{\partial \vec{E}_0}{\partial t} \cdot \vec{E}_1] + \mu [\vec{H}_0 \cdot \frac{\partial \vec{H}_1}{\partial t} + \frac{\partial \vec{H}_0}{\partial t} \cdot \vec{H}_1]) dv$$

$$= (\text{using } (1.2), (1.3)) =$$

$$= \iiint\limits_{\Omega} (\vec{E}_0 \cdot \text{curl } \vec{H}_1 - \text{curl } \vec{E}_0 \cdot \vec{H}_1 + \text{curl } \vec{H}_0 \cdot \vec{E}_1 - \vec{H}_0 \cdot \text{curl } \vec{E}_1) dv$$

$$= (\text{using div } (E \times H) = \text{curl } \vec{E} \cdot \vec{H} - \vec{E} \cdot \text{curl } \vec{H})$$

$$= - \iiint\limits_{\Omega} [\text{div}(\vec{E}_0 \times \vec{H}_1) + \text{div}(\vec{E}_1 \times \vec{H}_0)] dv$$

$$= - \iint\limits_{\Gamma} (\vec{E}_0 \times \vec{H}_1 + \vec{E}_1 \times \vec{H}_0) \cdot \nu ds = (\text{using } (1.7))$$

$$- \iint\limits_{\Gamma} (\vec{E}_{0\tau} \times \vec{H}_{1\tau} + E_{0\tau} \times \vec{H}_{1\nu} + \vec{E}_{1\tau} \times \vec{H}_{0\tau} + \vec{E}_{1\tau} \times \vec{H}_{0\nu}) \cdot \nu ds$$

$$= - \iint_\Gamma (\vec{E}_{0\tau} \times \vec{H}_{1\tau} + \vec{E}_{1\tau} \times \vec{H}_{0\tau}) \cdot \nu ds$$

$$= \text{(using (1.8) and noting that } J \equiv 0 \text{ for } \vec{E}_0, \vec{H}_0)$$

$$= - \iint_\Gamma (\vec{E}_{0\tau} \cdot \vec{J}) ds . \qquad (1.10)$$

If we go through the same computation with $\vec{E}_0, \vec{H}_0, \vec{E}_1, \vec{H}_1$ both replaced by the same $\vec{E}, \vec{H}$ satisfying (1.2)-(1.5), (1.7), (1.8) we find that

$$\frac{d\vec{E}}{dt} = - \iint_\Gamma (\vec{E} \times \vec{H}) \cdot \vec{\nu} ds = - \iint_\Gamma \vec{E}_\tau \cdot \vec{J} ds . \qquad (1.11)$$

For $\vec{J} \equiv 0$ generalized solutions of (1.2)-(1.5), (1.7), (1.8) can be discussed in the general context of partial differential equations and strongly continuous semigroups. The generator

$$A(\vec{E}, \vec{H}) = \left(\frac{1}{\epsilon} \text{curl } \vec{H}, - \frac{1}{\mu} \text{curl } \vec{E} \right) \qquad (1.12)$$

with domain consisting of $\vec{E}, \vec{H}$ in the Sobolev space $H^1_{E,d}(\Omega)(=H_{E,d}(\Omega) \cap H^1(\Omega))$ having zero divergence and satisfying (cf. (1.7), (1.8))

$$\vec{E}_\nu \big|_\Gamma = 0, \quad \vec{H}_\tau \big|_\Gamma = 0 , \qquad (1.13)$$

is antisymmetric and generates a group of isometries in $H_{E,d}(\Omega)$. (See [32], [33], [34] for related work.) Sufficient conditions on $\vec{J}$ so that solutions of the inhomogeneous system (1.2)-(1.5), (1.7), (1.8) lie in $H_{E,d}(\Omega)$ and are strongly continuous there may be obtained much as in [18], [19] but it is not easy to specify necessary and sufficient conditions. Indeed, this is already difficult for the much simpler, but related, wave equation

$$\mu\epsilon \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

with boundary forcing terms. We will make some comments related to this in Section 6.

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2. CONTROL PROBLEMS IN A CYLINDRICAL REGION

The main point in this paper is to study the question of controllability of the electromagnetic field $\vec{E}, \vec{H}$ by means of the boundary current $\vec{J} = \vec{J}_\tau$. By controllability we mean the possibility of transferring an initial field $\vec{E}(x,y,z,0)$, $\vec{H}(x,y,z,0)$ ∈ $H_{E,d}(\Omega)$, given at time $t = 0$, to a prescribed terminal field $\vec{E}(x,y,z,T)$, $\vec{H}(x,y,z,T)$ ∈ $H_{E,d}(\Omega)$, specified at $t = T > 0$, by means of a suitable control current $\vec{J}(x,y,z,t)$ defined for (x,y,z) ∈ Γ, t ∈ $[0,T]$. Because the homogeneous Maxwell equations correspond to a group of isometries in $H_{E,d}(\Omega)$, it is enough to consider the special case wherein

$$\vec{E}(x,y,z,0) \equiv 0 , \tag{2.1}$$

$$\vec{H}(x,y,z,0) \equiv 0 . \tag{2.2}$$

For a given space, J, of admissible control currents $\vec{J}(x,y,z,t) = \vec{J}_\tau(x,y,z,t)$ defined on $\Gamma \times [0,T]$ we define the reachable set $R(T,J)$ to be the subspace of $H_{E,d}(\Omega)$ consisting of states reachable from the zero initial state using controls $\vec{J}$ ∈ J. Following earlier definitions ([8], [26]), our system is approximately controllable in time T if $R(T,J)$ is dense in $H_{E,d}(\Omega)$ and exactly controllable in time T if $R(T,J) = H_{E,d}(\Omega)$ (or some precisely designated subspace of $H_{E,d}(\Omega)$).

At this writing we are not able to discuss the general three dimensional problem wherein the vector fields $\vec{E}$ and $\vec{H}$ are unrestricted, except as stipulated heretofore, and Ω has a general geometry. We hope in later work to consider at least some three dimensional cases which arise for special domains Ω. But for now we must content ourselves with the case in which Ω is a cylinder:

$$\Omega = R \times (-\infty,\infty) = \{(x,y,z)|(x,y) \in R \subset R^2, z \text{ real}\}$$

where R is an open connected region in R^2 with piecewise smooth boundary B. Thus

$$\partial\Omega = \partial R \times (-\infty,\infty) = B \times (-\infty,\infty) .$$

Even here we can give results only for special two dimensional regions R.

The two dimensional problem in the cylinder $\Omega = R \times (-\infty,\infty)$ occurs when we confine attention to fields

$$\vec{E} = \vec{E}(x,y,t), \vec{H} = \vec{H}(x,y,t)$$

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which do not depend on the coordinate z corresponding to the axial, or longitudinal, direction of the cylinder. (Note that this is not at all the same thing as requiring that E_z, H_z, the field components in the z direction, should be zero.) We correspondingly consider only control currents

$$\vec{J} = \vec{J}(x,y,t)$$

which do not depend upon z.

Of course the energy E in Ω is infinite under the above circumstances if $\vec{E}$, $\vec{H}$ are not identically zero. We redefine E to be the energy per unit length of cylinder:

$$E(t) = \tfrac{1}{2} \iint_R (\epsilon |\vec{E}(x,y,t)|^2 + \mu |\vec{H}(x,y,t)|^2)dxdy . \qquad (2.3)$$

The space $H_{E,d}(\Omega)$ is now replaced by $H_{E,d}(R)$. Because

$$\frac{\partial E_z(x,y,t)}{\partial z} \equiv 0, \quad \frac{\partial H_z(x,y,t)}{\partial z} \equiv 0$$

we have

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}, \quad \operatorname{div} \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} . \qquad (2.4)$$

The curl expressions simplify to

$$\operatorname{curl} \vec{E} = \left(\frac{\partial E_z}{\partial y}, -\frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) ,$$

$$\operatorname{curl} \vec{H} = \left(\frac{\partial H_z}{\partial y}, -\frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) ,$$

so that the equations (1.2), (1.3) become

$$(i) \quad \epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} \qquad (iv) \quad \mu \frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y}$$

$$(ii) \quad \epsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} \qquad (v) \quad \mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} \qquad (2.5)$$

$$(iii) \quad \epsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \qquad (vi) \quad \mu \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}$$

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It is clear from (2.5), (i)-(vi), that if $\vec{E}(x,y,0)$, $\vec{H}(x,y,0)$ are given, then the subsequent evolution of $E_z(x,y,t)$, $H_z(x,y,t)$ determine all of the other components. As for these components themselves, differentiating (2.5) (iii) and (2.5) (vi) with respect to t and then substituting (2.5) (iv), (v) and (2.5) (i), (ii) into the respectively resulting expressions, we obtain the familiar wave equations

$$\mu\epsilon \frac{\partial^2 E_z}{\partial t^2} = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} , \qquad (2.6)$$

$$\mu\epsilon \frac{\partial^2 H_z}{\partial t^2} = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} , \qquad (2.7)$$

valid for $(x,y) \in R$, $t \in [0,\infty)$, provided E_z, H_z have enough derivatives, or provided the equations are interpreted in the distributional sense. Assuming the initial states $\vec{E}(x,y,0)$, $\vec{H}(x,y,0)$ are divergence-free, we compute (cf. (2.4))

$$\epsilon \frac{\partial}{\partial t} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) = (\text{using } (2.5) \text{ (i), (ii)})$$

$$\epsilon \left(\frac{\partial^2 H_z}{\partial x \partial y} - \frac{\partial^2 H_z}{\partial y \partial x} \right) = 0$$

and similarly

$$\mu \frac{\partial}{\partial t} \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right) = 0$$

and we conclude that the fields remain divergence-free for all time.

Suppose, then, that divergence-free initial states $\vec{E}(x,y,0)$, $\vec{H}(x,y,0)$ are given. Then $E_z(x,y,0)$, $H_z(x,y,0)$ are known and (2.5) (iii), (vi) determine $\frac{\partial E_z}{\partial t}(x,y,0)$ and $\frac{\partial H_z}{\partial t}(x,y,0)$. If (2.6), (2.7) are then solved with these initial conditions, and appropriate boundary conditions, the complete solution of Maxwell's equations (2.5) (i)-(vi), can be obtained by integrating (2.5) (i), (ii), (iv), (v). Thus it is enough to work with (2.6), (2.7), and it should be noted that the divergence condition does not have any bearing on E_z, H_z; it can be ignored henceforth.
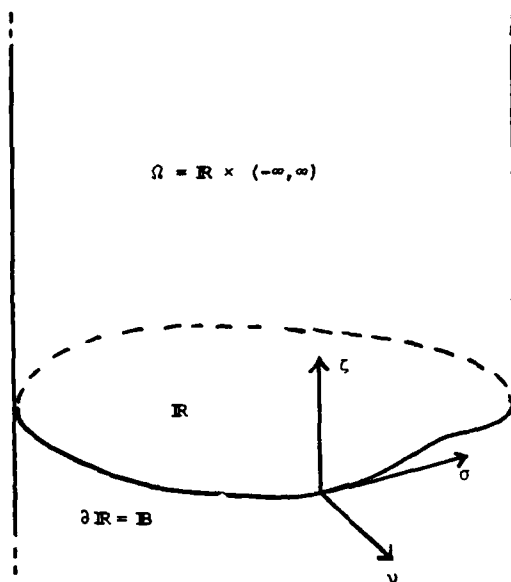
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$$\Omega = \mathbb{R} \times (-\infty, \infty)$$

Figure 1. The Region $\mathbb{R}$

It is important to recast the boundary conditions (1.7), (1.8) so that they provide boundary conditions for (2.6), (2.7). We ask the reader to consult Figure 1, where the region $\mathbb{R}$ with boundary $\partial \mathbb{R} = B$ is shown. At a point $(x,y) \in B$ we let $\vec{\nu} = \vec{\nu}(x,y)$ denote the unit exterior normal to B and we let $\vec{\sigma} = \vec{\sigma}(x,y)$ denote the positively oriented unit tangent vector to B there. With $\vec{\zeta}$, the unit vector in the positive z direction, $\vec{\nu}, \vec{\sigma}, \vec{\zeta}$ form a positively oriented orthogonal triple of unit vectors. Given an arbitrary vector w we can decompose it as

$$\vec{w} = (w_{\nu}, w_{\sigma}, w_{\zeta}(=w_z)) ,$$

$$|\vec{w}|^2 = w_{\nu}^2 + w_{\sigma}^2 + w_z^2 .$$

The tangential part of $\vec{H}$, which we have designated as $\vec{H}_{\tau}$, may now be represented as

$$\vec{H}_{\tau} = H_z \vec{\zeta} + H_{\sigma} \vec{\sigma} \tag{2.8}$$

and the current $\vec{J} = \vec{J}_{\tau}$ may likewise be represented as

$$\vec{J}_{\tau} = J_z \vec{\zeta} + J_{\sigma} \vec{\sigma} .$$

Then

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$$\vec{\nu} \times \vec{J} = \vec{\nu} \times \vec{J}_\tau = \vec{\nu} \times (J_z \vec{\xi} + J_\sigma \vec{\sigma}) = -J_z \vec{\sigma} + J_\sigma \vec{\xi} \qquad (2.9)$$

Combining (1.8), (2.8), (2.9) we see that on **B**

$$H_z(x,y,t) = J_\sigma(x,y,t) , \qquad (2.10)$$

$$H_\sigma(x,y,t) = -J_z(x,y,t) . \qquad (2.11)$$

Represent $\vec{\nu}$, $\vec{\sigma}$ as

$$\vec{\nu} = \nu_x \vec{\xi} + \nu_y \vec{\eta} , \qquad (2.12)$$

$$\vec{\sigma} = \sigma_x \vec{\xi} + \sigma_y \vec{\eta} = -\nu_y \vec{\xi} + \nu_x \vec{\eta} . \qquad (2.13)$$

Then compute

$$\frac{\partial E_z}{\partial \nu} = \frac{\partial E_z}{\partial x} \nu_x + \frac{\partial E_z}{\partial y} \nu_y = \text{(using (1.3)} , \qquad (2.13)$$

$$= \mu \frac{\partial H_y}{\partial t} \sigma_y + \mu \frac{\partial H_x}{\partial t} \sigma_x = \mu \frac{\partial H_\sigma}{\partial t}$$

$$= \text{(using (2.11))} = - \frac{\partial J_z}{\partial t} . \qquad (2.14)$$

The equations (2.10), (2.14) provide the needed boundary conditions for (2.6), (2.7) respectively. For H_z we have the Dirichlet-type boundary condition (2.10) while for E_z we have the Neumann-type boundary condition (2.14). If we let

$$\vec{U}(x,y,t) = \frac{\partial \vec{J}}{\partial t} (x,y,t) ,$$

$$\vec{U} = \vec{U}_\tau = U_\sigma \vec{\sigma} + U_z \vec{\xi} ,$$

and differentiate (2.10), we have the more symmetric form

$$\frac{\partial H_z}{\partial t} (x,y,t) = U_\sigma(x,y,t), \quad \frac{\partial E_z}{\partial \nu} = -U_z(x,y,t), \quad (x,y) \in B . \qquad (2.15)$$

We complete this section by discussing the question of expression of the energy per unit cylinder length, (2.3), solely in terms of H_z and E_z.

We consider the equations (2.6), (2.7) with homogeneous boundary conditions

$$\frac{\partial H_z}{\partial t} (x,y,t) = 0, \quad \frac{\partial E_z}{\partial \nu} (x,y,t) = 0, \quad (x,y) \in B .$$

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We use the symbol Δ for the Laplacian:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ .$$

Initially we take H_z, E_z to lie in the Sobolev space $H^2(R)$. This space must be decomposed in order to attach a meaning to Δ^{-1}.

The boundary condition for H_z may be rewritten as

$$H_z(x,y,t) = h(x,y), \quad (x,y) \in B \ ,$$

where, by the trace theorem, $h \in H^{3/2}(B)$. Then we can write

$$H_z(x,y,t) = \hat{H}_z(x,y,t) + \tilde{H}_z(x,y)$$

where $\tilde{H}_z(x,y)$ is the solution of

$$\Delta \tilde{H}_z(x,y) = 0, \quad \tilde{H}_z(x,y) = h(x,y), \quad (x,y) \in B$$

and

$$\hat{H}_z(x,y,t) = 0, \quad (x,y) \in B \ .$$

The inverse Laplacian Δ^{-1} is well defined on the functions $\hat{H}_z$. For E_z we may write

$$E_z(x,y,t) = \hat{E}_z(x,y,t) + \tilde{E}_z(t)$$

where $\tilde{E}_z$, as indicated, is constant with respect to $(x,y) \in R$ and

$$\int_B \hat{E}_z(x,y,t) ds = 0 \ .$$

It is well known that Δ^{-1} is well defined on the functions $\hat{E}_z$.

We proceed first on the assumption that

$$H_z(x,y,t) = \hat{H}_z(x,y,t), \quad E_z(x,y,t) = \hat{E}_z(x,y,t) \ .$$

We form new solutions of (2.6), (2.7) by setting

$$\mu \frac{\partial G_z}{\partial t} = -E_z, \quad \epsilon \frac{\partial F_z}{\partial t} = H_z \ ,$$

$$G_z = \epsilon \Delta^{-1} \frac{\partial E_z}{\partial t}, \quad F_z = \mu \Delta^{-1} \frac{\partial H_z}{\partial t} \ .$$

We then determine G_x, G_y, F_x, F_y, using the equations (2.5) with $\hat{G}$ replacing $\hat{H}$, $\hat{F}$ replacing $\hat{E}$, so that $\hat{F}$ and $\hat{G}$ satisfy Maxwell's equations:

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$$\mu \, \frac{\partial \vec{G}}{\partial t} = -\text{curl} \, \vec{F} \ ,$$

$$\epsilon \, \frac{\partial \vec{F}}{\partial t} = \text{curl} \, \vec{G} \ .$$

It will then be found that

$$\vec{E} = \text{curl} \, \vec{F}, \quad \vec{H} = \text{curl} \, \vec{G} \ .$$

Following this, (2.3) can be written as

$$E(t) = \tfrac{1}{2} \iint\limits_{R} (\epsilon |\text{curl} \, \vec{F}|^2 + \mu |\text{curl} \, \vec{G}|^2) dx dy$$

$$= \tfrac{1}{2} \iint\limits_{R} \epsilon \Big[\Big(\frac{\partial F_z}{\partial x}\Big)^2 + \Big(\frac{\partial F_z}{\partial y}\Big)^2 + \Big(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\Big)^2 \Big]$$

$$+ \mu \Big[\Big(\frac{\partial G_z}{\partial x}\Big)^2 + \Big(\frac{\partial G_z}{\partial y}\Big)^2 + \Big(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\Big)^2 \Big] dx dy \qquad (2.16)$$

Then from (2.16) we have

$$E(t) = \tfrac{1}{2} \iint\limits_{R} \Big\{ \epsilon \Big[\Big(\frac{\partial F_z}{\partial x}\Big)^2 + \Big(\frac{\partial F_z}{\partial y}\Big)^2 + \Big(\mu \frac{\partial G_z}{\partial t}\Big)^2 \Big]$$

$$+ \mu \Big[\Big(\frac{\partial G_z}{\partial x}\Big)^2 + \Big(\frac{\partial G_z}{\partial y}\Big)^2 + \Big(\epsilon \frac{\partial F_z}{\partial t}\Big)^2 \Big] \Big\} \, dx dy$$

$$= \tfrac{1}{2} \iint\limits_{R} \epsilon \Big[\Big(\frac{\partial F_z}{\partial x}\Big)^2 + \Big(\frac{\partial F_z}{\partial y}\Big)^2 + (E_z)^2 \Big]$$

$$+ \mu \Big[\Big(\frac{\partial G_z}{\partial x}\Big)^2 + \Big(\frac{\partial G_z}{\partial y}\Big)^2 + (H_z)^2 \Big] dx dy$$

Now consider the quadratic form (for $E_z = \hat{E}_z$)

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$$\left(\frac{\partial E_z}{\partial t}, -\Delta^{-1}\frac{\partial E_z}{\partial t}\right) = \left(\mu\frac{\partial^2 G_z}{\partial t^2}, -\Delta^{-1}\frac{\partial^2 G_z}{\partial t^2}\right)$$

= (since G_z satisfies the wave equation $\mu\epsilon\dfrac{\partial^2 G_z}{\partial t^2} = \Delta G_z$

and the boundary conditions $G_z(x,y,t) = 0$, $(x,y) \in B$) =

$$\frac{1}{\mu\epsilon^2}(-\Delta G_z, G_z) = \frac{1}{\mu\epsilon^2}\left[\left(\frac{\partial G_z}{\partial x}\right)^2 + \left(\frac{\partial G_z}{\partial y}\right)^2\right] .$$

Similarly

$$\left(\frac{\partial H_z}{\partial t}, -\Delta^{-1}\frac{\partial H_z}{\partial t}\right) = \frac{1}{\epsilon\mu^2}\left[\left(\frac{\partial F_z}{\partial x}\right)^2 + \left(\frac{\partial F_z}{\partial y}\right)^2\right]$$

from which it follows that

$$E(t) = \tfrac{1}{2}\iint_R \left\{(\mu\epsilon)^2\left[\left(\frac{\partial E_z}{\partial t}, -\Delta^{-1}\frac{\partial E_z}{\partial t}\right) + \left(\frac{\partial H_z}{\partial t}, -\Delta^{-1}\frac{\partial H_z}{\partial t}\right)\right] + \epsilon(E_z)^2 + \mu(H_z)^2\right\}dxdy .$$

¶ finite states – a fact which will be very careful later.

It is necessary to modify this expression for general E_z, H_z. We begin with

$$E_z(x,y,t) = \widetilde{E}_z(t) .$$

The only possible solutions of the wave equation (2.6) satisfying $\dfrac{\partial E_z}{\partial\nu}\Big|_B = 0$ and having

this form are

$$E_z(x,y,t) = e_0 + e_1 t$$

where e_0 and e_1 are constants. (Such solutions are consistent with a constant boundary

current J for which $J_\sigma \equiv 0$.) The corresponding E_x, E_y, H_z are zero but

$$\epsilon e_1 = \epsilon\frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} .$$

It is not possible to express this quantity in terms of E_z itself or H_z. It is better

to leave it in the form $\epsilon\dfrac{\partial E_z}{\partial t}$. Solutions of Maxwell's equations with E_z having this

form have energy expressible as a quadratic form in E_z and $\dfrac{\partial E_z}{\partial t}$.

Next we consider $H_z = \widetilde{H}_z$ as described earlier. Such a solution is consistent with a boundary current for which $J_\tau = 0$, constant with respect to time but possibly varying with $(x,y) \in B$. We may take H_x, H_y, E_z all zero. However,

$$\epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, \quad \epsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}$$

so we may not assume that E_x and E_y are equal to zero. The energy associated with solutions of this type is expressible in terms of

$$\iint_R \left[\left(\frac{\partial H_z}{\partial x}\right)^2 + \left(\frac{\partial H_z}{\partial y}\right)^2\right] dx dy$$

if integration with respect to t is permitted. In the sequel we will not explicitly consider the timewise linear electric fields satisfying the above equations.

We see then that a norm involving only E_z and H_z and compatible with the energy (2.3) may be expressed as

$$\mathbf{|}(E_z, H_z)\mathbf{|}^2 = \iint_R (\mu\epsilon)^2 \left[\left(\frac{\partial \hat{E}_z}{\partial t}, -\Delta^{-1} \frac{\partial \hat{E}_z}{\partial t}\right) + \left(\frac{\partial \hat{H}_z}{\partial t}, -\Delta^{-1} \frac{\partial \hat{H}_z}{\partial t}\right)\right] + \epsilon(\hat{E}_z)^2 + \mu(\hat{H}_z)^2$$

$$\rho_0(\widetilde{E}_z)^2 + \rho_1\left(\frac{\partial \widetilde{E}_z}{\partial t}\right)^2 + \sigma_0\left(\frac{\partial \widetilde{H}_z}{\partial x}\right)^2 + \sigma_1\left(\frac{\partial \widetilde{H}_z}{\partial y}\right)^2\right] dx dy \qquad (2.17)$$

where $\rho_0, \rho_1, \sigma_0, \sigma,$ are positive numbers. It will be seen that this is a weaker norm than the one associated with a pair of wave equations, viz.:

$$\mathbf{|}(E_z, H_z)\mathbf{|}^2 = \iint_R \left\{\mu\epsilon\left[\left(\frac{\partial E_z}{\partial t}\right)^2 + \left(\frac{\partial H_z}{\partial t}\right)^2\right] + \mathbf{|}\nabla E_z\mathbf{|}^2 + \mathbf{|}\nabla H_z\mathbf{|}^2\right\} dx dy . \qquad (2.18)$$

We will denote the Hilbert space of states E_z, H_z, $\frac{\partial E_z}{\partial t}$, $\frac{\partial H_z}{\partial t}$ lying in $H^1(R)$, $H^1(R)$, $L^2(R)$, $L^2(R)$, respectively, by $\hat{H}$. This space will be very convenient for use in the remainder of this paper. In some cases we will add boundary conditions to the specification of H, the space with norm $\mathbf{|} \mathbf{|}$, without changing the symbol, to correspond to an agreed specification of the states in $\hat{H}$ by similar boundary conditions.

3. SOME CONTROL CONFIGURATIONS

We describe here two possible realizations of the control problem which we have posed and indicate why we have chosen the mathematically more interesting (i.e., more difficult) one to work with in this paper.

Let us assume that $\Gamma = \partial\Omega = B \times (-\infty, \infty)$ is covered by one or more layers of conducting bars, arranged in rows as shown in Figure 3.1. In the case of a single layer of conducting bars shown in Figure 2(b), the bars are arranged so that they make an angle θ, $0 < |\theta| < \frac{\pi}{2}$, with the vector $\vec{\sigma}$ (cf. Figure 1), while in the double layer case (Figure 2(a)) they are arranged so that the bars in the second layer make an angle ψ, $0 < |\psi| < \frac{\pi}{2}$, $\psi \neq \theta$, with the vector $\vec{\sigma}$. The current in any row of bars parallel to the z-axis is independent of z; i.e., constant for all bars in that row. As we consider successively smaller bars we obtain, as an idealization, the boundary current vector
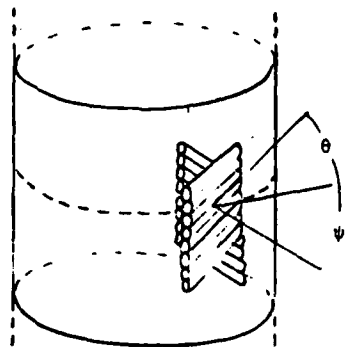
Figure 2(a). Double Layer Control
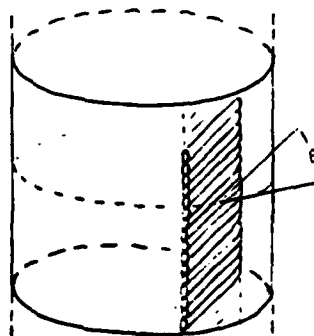
Figure 2(b). Single Layer Control

$$\vec{J}(x,y,t) = J(x,y,t)(\cos\theta\vec{\sigma} + \sin\theta\vec{\zeta}) \qquad (3.1)$$

in the single layer case, $J(x,y,t)$ denoting the current strength with the sign determined so that J positive yields a positive current component in the $\vec{\sigma}$ direction. The corresponding formula in the double layer case is

$$\vec{J}(x,y,t) = J_1(x,y,t)(\cos\theta\vec{\sigma} + \sin\theta\vec{\zeta})$$
$$+ J_2(x,y,t)(\cos\psi\vec{\sigma} + \sin\psi\vec{\zeta}) . \qquad (3.2)$$

The current components are, in the single layer case

$$J_\sigma(x,y,t) = J(x,y,t)\cos\theta ,$$

$$J_z(x,y,t) = J(x,y,t)\sin\theta ,$$

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and in the double layer case,

$$\begin{pmatrix} J_\sigma(x,y,t) \\ J_z(x,y,t) \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\psi \\ \sin\theta & \sin\psi \end{pmatrix} \begin{pmatrix} J_1(x,y,t) \\ J_2(x,y,t) \end{pmatrix} . \tag{3.3}$$

The determinant of the matrix in (3.3) is $\sin(\psi - \theta) \neq 0$ if $\psi \neq \theta$ in the range $0 < |\theta| < \frac{\pi}{2}$, $0 < |\psi| < \frac{\pi}{2}$. Thus in the double layer case J_σ and J_z are independent if J_1 and J_2 are independent while in the single layer case J_σ and J_z are fixed non-zero multiples of each other.

The double layer case is easily disposed of in the light of earlier work on boundary control of the wave equation. Referring back to (2.10), (2.11) we now have, for $(x,y) \in B = \partial R$, $t \in [0,\infty)$,

$$\frac{\partial H_z}{\partial t}(x,y,t) = U_\sigma(x,y,t) = \cos\theta\, u_1(x,y,t) + \cos\psi\, u_2(x,y,t) ,$$

$$\frac{\partial E_z}{\partial \nu}(x,y,t) = -U_z(x,y,t) = -\sin\theta\, u_1(x,y,t) + \cos\psi\, u_2(x,y,t) ,$$

$$u_1(x,y,t) = \frac{\partial J_1}{\partial t}(x,y,t), \quad u_2(x,y,t) = \frac{\partial J_2}{\partial t}(x,y,t) .$$

Since U_σ and U_z are independent if u_1 and u_2 are, the control problem splits into two uncoupled wave-equation problems, one for E_z and one for H_z. These have been discussed thoroughly in [2], [3], [15], [16], [22], [23], [25] with affirmative controllability results for various control configurations and will not concern us further here.

In the remainder of this paper we study the single layer case. If we let

$$u(x,y,t) = \frac{\partial J}{\partial t}(x,y,t) \tag{3.4}$$

we now have the wave equations (2.6), (2.7) for E_z, H_z and the boundary conditions

$$\frac{\partial H_z}{\partial t}(x,y,t) = \cos\theta\, \frac{\partial J}{\partial t}(x,y,t) \equiv \alpha\, u(x,y,t) , \tag{3.5}$$

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$$\frac{\partial E_z}{\partial \nu}(x,y,t) = -\sin\theta \frac{\partial J}{\partial t}(x,y,t) \equiv \beta u(x,y,t) . \qquad (3.6)$$

The control problems for E_z and H_z are now coupled because the single control function, $u(x,y,t)$, appears in the boundary conditions for both E_z and H_z; we have to contol both systems simultaneously using the same control function.

If we rely on experience in a single space dimension, which has proved generally quite helpful in the control theory of a single wave equation, we are led to believe that systems like (2.6), (2.7), (3.5), (3.6) may, in fact, be controllable. Replacing $u(x,y,t)$ by $u_0(t)$, $u_1(t)$ and taking $0 < x < 1$, the one dimensional equations are, using variables v, w,

$$\rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0 , \qquad (3.7)$$

$$\frac{\partial v}{\partial t}(0,t) = \alpha u_0(t), \quad \frac{\partial v}{\partial t}(1,t) = \alpha u_1(t) , \qquad (3.8)$$

$$\rho \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0 \qquad (3.9)$$

$$\frac{\partial w}{\partial x}(0,t) = -\beta u_0(t), \quad \frac{\partial w}{\partial x}(1,t) = \beta u_1(t) \qquad (3.10)$$

(note that $-\frac{\partial w}{\partial x}$ corresponds to the exterior normal derivative at 0]. Letting

$$\tilde{v} = \frac{\partial v}{\partial x} \qquad (3.11)$$

$$\tilde{w} = \frac{\partial w}{\partial t} \qquad (3.12)$$

we find that

$$\rho \frac{\partial^2 \tilde{v}}{\partial t^2} - \frac{\partial^2 \tilde{v}}{\partial x^2} = 0 , \qquad (3.13)$$

and

$$\rho \frac{\partial^2 \tilde{w}}{\partial t^2} - \frac{\partial^2 \tilde{w}}{\partial x^2} = 0 . \qquad (3.14)$$

Differentiating (3.11) with respect to t and using (3.8) we have

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$$\frac{1}{\rho} \frac{\partial^2 v}{\partial x^2} (0,t) = \frac{1}{\rho} \frac{\partial \tilde{v}}{\partial x} (0,t) = \frac{\alpha}{\rho} u_0'(t) , \qquad (3.15)$$

$$\frac{1}{\rho} \frac{\partial^2 v}{\partial x^2} (1,t) = \frac{1}{\rho} \frac{\partial \tilde{v}}{\partial x} (1,t) = \frac{\alpha}{\rho} u_1'(t) , \qquad (3.16)$$

while differentiation of (3.12) along with (3.10) yields

$$\frac{\partial^2 w}{\partial t \partial x} (0,t) = \frac{\partial \tilde{w}}{\partial x} (0,t) = -\beta u_0'(t) , \qquad (3.17)$$

$$\frac{\partial^2 w}{\partial t \partial x} (1,t) = \frac{\partial \tilde{w}}{\partial x} (1,t) = \beta u_1'(t) . \qquad (3.18)$$

Combining (3.13) with (3.14), (3.15), (3.16), (3.17), (3.18), we see that $\beta \tilde{v} + \frac{\alpha}{\rho} \tilde{w}$, $\beta \tilde{v} - \frac{\alpha}{\rho} \tilde{w}$ both satisfy the wave equation and

$$\frac{\partial}{\partial x} (\beta \tilde{v} + \frac{\alpha}{\rho} \tilde{w})(0,t) = 0, \quad \frac{\partial}{\partial x} (\beta \tilde{v} + \frac{\alpha}{\rho} \tilde{w})(1,t) = \frac{2\alpha\beta}{\rho} u_1'(1) ,$$

$$\frac{\partial}{\partial x} (\beta \tilde{v} - \frac{\alpha}{\rho} \tilde{w})(0,t) = \frac{2\alpha\beta}{\rho} u_0'(t), \quad \frac{\partial}{\partial x} (\beta \tilde{v} - \frac{\alpha}{\rho} \tilde{w})(1,t) = 0 .$$

Thus the control problems for $\beta \tilde{v} + \frac{\alpha}{\rho} \tilde{w}$ and $\beta \tilde{v} - \frac{\alpha}{\rho} \tilde{w}$ are both of Neumann type and are uncoupled. Affirmative controllability results are then available from [20], [21], [24].

If we replace $u_0(t)$ (or $u_1(t)$ by 0 in the above, then $\beta \tilde{v} - \frac{\alpha}{\rho} \tilde{w}$ (or $\beta \tilde{v} + \frac{\alpha}{\rho} \tilde{w}$) will become completely uncontrollable and our original system must therefore be uncontrollable. This result at first seems to predict failure for the enterprize which we now undertake for the two dimensional case.

4. APPROXIMATE BOUNDARY CONTROLLABILITY

By a simple change of scale in the t variable, and renaming of the independent variables, we may assume that the system of interest is

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \ , \qquad (4.1)$$

$$\left. \begin{array}{l} \end{array} \right\} \quad t > 0 \ , \quad (x,y) \in R \ ,$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \ , \qquad (4.2)$$

with boundary conditions

$$\frac{\partial v}{\partial t} (x,y,t) = \alpha u(x,y,t) \qquad (4.3)$$

$$\left. \begin{array}{l} \end{array} \right\} \quad t > 0 \ , \quad (x,y) \in B = \partial \Omega$$

$$\frac{\partial w}{\partial v} (x,y,t) = \beta u(x,y,t) \qquad (4.4)$$

We will not, in general, assume that u(x,y,t) can be selected at will for all values of (x,y,t) shown. More on this later.

Because the system is time reversible, it *is sufficient to analyze controllability in* terms of control from the zero initial state

$$v(x,y,0) = \frac{\partial v}{\partial t} (x,y,0) = 0 \ , \qquad (4.5)$$

$$\left. \begin{array}{l} \end{array} \right\} \quad (x,y) \in R \ ,$$

$$w(x,y,0) = \frac{\partial w}{\partial t} (x,y,0) = 0 \ , \qquad (4.6)$$

to a final state

$$v(x,y,T) = v_0(x,y), \quad \frac{\partial v}{\partial t} (x,y,T) = v_1(x,y) \qquad (4.7)$$

$$\left. \begin{array}{l} \end{array} \right\} \quad (x,y) \in R \ .$$

$$w(x,y,T) = w_0(x,y), \quad \frac{\partial w}{\partial t} (x,y,T) = w_1(x,y) \qquad (4.8)$$

We have noted in Section 2 that the $|\ |$- finite states are dense in the $\|\ \|$- finite states. In the present context this means that we can work with the Hilbert space of states v, $\frac{\partial v}{\partial t}$, w, $\frac{\partial w}{\partial t}$ with the inner product

$$((v, \tfrac{\partial v}{\partial t}, w, \tfrac{\partial w}{\partial t}); (\tilde{v}, \tfrac{\partial \tilde{v}}{\partial t}, \tilde{w}, \tfrac{\partial \tilde{w}}{\partial t}))$$

$$= \iint_R [\tfrac{\partial v}{\partial t}\tfrac{\partial \tilde{v}}{\partial t} + \tfrac{\partial w}{\partial t}\tfrac{\partial \tilde{w}}{\partial t} + \tfrac{\partial v}{\partial x}\tfrac{\partial \tilde{v}}{\partial x} + \tfrac{\partial w}{\partial x}\tfrac{\partial \tilde{w}}{\partial x} + \tfrac{\partial v}{\partial y}\tfrac{\partial \tilde{v}}{\partial y} + \tfrac{\partial w}{\partial y}\tfrac{\partial \tilde{w}}{\partial y}]dxdy , \qquad (4.9)$$

a space which we will refer to as $\hat{H}$. The norm is $|\ |$ (cf. (2.18)) with $\mu\epsilon = 1$. As we have indicated, this is a dense subspace of H, the Hilbert space obtained by use of the norm $\|\ \|$ (cf. (2.17)).

The final states (4.7), (4.8) are not quite arbitrary in $\hat{H}$ if the control u is restricted so that its support is contained in a proper relatively closed subset $B_1 \subset B$. Since the condition

$$\frac{\partial v}{\partial t} (x,y,t) = \alpha\, u(x,y,t), \quad (x,y) \in B$$

applies, we may as well adjoin the additional condition

$$v_0(x,y) = 0, \quad (x,y) \in B - B_1 \equiv \mathbb{B}_0 . \qquad (4.10)$$

The trace theorem ([1], [19]) assures us that this describes a closed subspace of $\hat{H}$, which we will call $\hat{H}_1$. The only restriction on $\hat{H}_1$ is (4.10); v_0 is permitted to have arbitrary values in $H^{1/2}(\mathbb{B}_1)$ and w_0, w_1 are unrestricted in $H^1(\mathbb{B})$, $H^0(\mathbb{R}) = L^2(\mathbb{R})$, respectively.

Let U be a given space of admissible control functions, about which we will shortly have more to say. For each control $u \in U$ we assume the existence of a unique solution v_u, w_u of (4.1)-(4.6) for $t > 0$, $(x,y) \in \mathbb{R}$. Very general sufficient conditions for this to be the case are given in [19]. We define the reachable set at time T, $R(U,T)$, to be the set of all final states $v_u(x,y,T)$, $\frac{\partial v_u}{\partial t} (x,y,T)$, $w_u(x,y,T)$, $\frac{\partial w_u}{\partial t} (x,y,T)$ which may be realized in this way. The set $R(U,T)$ is a subspace of $\hat{H}_1$ if U is a linear space, which we will assume, and our system is approximately controllable in time T if $R(U,T)$ is dense in $\hat{H}_1$ (then $R(U,T)$ is also dense in H because $\|\ \|$ is a weaker norm than $|\ |$ and $\hat{H}_1$ is dense in H). Evidently $R(U,T)$ is dense in $\hat{H}_1$ just in case, given an

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arbitrary state $(\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)$ in $\hat{H}_1$,

$$\{((v_u(x,y,T), \frac{\partial v_u}{\partial t}(x,y,T), w_u(x,y,T), \frac{\partial w_u}{\partial t}(x,y,T)), (\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)) = 0 \ ,$$

$$u \in U\} \implies (\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1) = 0 \ . \tag{4.11}$$

Let $\tilde{v}(x,y,t)$, $\tilde{w}(x,y,t)$ be the unique solution of (4.1), (4.2) satisfying the terminal conditions at time T:

$$\tilde{v}(x,y,T) = \tilde{v}_0, \quad \frac{\partial \tilde{v}}{\partial t}(x,y,T) = \tilde{v}_1, \quad \tilde{w}(x,y,T) = \tilde{w}_0, \quad \frac{\partial \tilde{w}}{\partial t}(x,y,T) = \tilde{w}_1 \ , \tag{4.12}$$

and the homogeneous boundary conditions

$$\left. \begin{aligned} \frac{\partial \tilde{v}}{\partial t}(x,y,t) &= 0 \ , \\[2em] \frac{\partial \tilde{w}}{\partial \nu}(x,y,t) &= 0 \ , \end{aligned} \right\} \quad (x,y) \in B, \quad t > 0 \ . \tag{4.13} \tag{4.14}$$

Computing the quantity

$$\frac{d}{dt} ((v_u(x,y,t), \frac{\partial v_u}{\partial t}(x,y,t), w_u(x,y,t), \frac{\partial w_u}{\partial t}(x,y,t)) \ ,$$

$$(\tilde{v}(x,y,t), \frac{\partial \tilde{v}}{\partial t}(x,y,t), \tilde{w}(x,y,t), \frac{\partial \tilde{w}}{\partial t}(x,y,t))) \ ,$$

using familiar duality theorems involving the Laplacian and integrating from 0 to T (see [22], [23], [26] for details in the case of a single wave equation) we see that

$$((v_u(x,y,T), \frac{\partial v_u}{\partial t}(x,y,T), w_u(x,y,T), \frac{\partial w_u}{\partial t}(x,y,T)), (\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1))$$

$$= \int_0^T \int_B [\frac{\partial \tilde{v}}{\partial t}(x,y,t) \frac{\partial v_u}{\partial \nu}(x,y,t) + \frac{\partial \tilde{v}}{\partial \nu}(x,y,t) \frac{\partial v_u}{\partial t}(x,y,t)$$

$$+ \frac{\partial \tilde{w}}{\partial t}(x,y,t) \frac{\partial w_u}{\partial \nu}(x,y,t) + \frac{\partial \tilde{w}}{\partial \nu}(x,y,t) \frac{\partial w_u}{\partial t}(x,y,t)] ds dt \ . \tag{4.15}$$

Then using the boundary conditions (4.3), (4.4), (4.13), (4.14) we see that the above

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reduces to

$$\int_0^T \int_B [\alpha \frac{\partial \tilde{v}}{\partial \nu}(x,y,t) + \beta \frac{\partial \tilde{w}}{\partial t}(x,y,t)]u(x,y,t)dsdt . \qquad (4.16)$$

If, as discussed above, we suppose that B has the disjoint decomposition

$$B = B_0 \cup B_1 ,$$

with B_1 relatively open in B, and that $u(x,y,t) \equiv 0$, $(x,y) \in B_0$ while on B_1 u is

unrestricted save for the specification of the admissible space (e.g., we might take

$$U = C(B_1 \times [0,T]), \quad U = L^2(B_1 \times [0,T]) , \qquad (4.17)$$

or any of many other possibilities), and if we suppose the first equation in (4.11) to

hold, we conclude that (4.16) vanishes for all $u \in U$. We know from the trace theorem

([1], [19]) that the partial derivatives

$$\frac{\partial \tilde{v}}{\partial t}, \frac{\partial \tilde{v}}{\partial \nu}, \frac{\partial \tilde{w}}{\partial t}, \frac{\partial \tilde{w}}{\partial \nu} ,$$

restricted to B, all lie in $H^{1/2}(B)$ for $t \in [0,T]$ and vary, with respect to the norm

in that space, continuously with respect to t, i.e. they lie in $C(H^{1/2}(B); [0,T])$. We

suppose, as is the case for (4.17), e.g., that U includes a total subspace of the dual

space of $C(H^{1/2}(B_1);[0,T])$. Then the fact that (4.17) is zero for all $u \in U$ implies

$$\alpha \frac{\partial \tilde{v}}{\partial \nu}(x,y,t) + \beta \frac{\partial \tilde{w}}{\partial t}(x,y,t) = 0, \quad (x,y) \in B_1, \quad t \in [0,T] . \qquad (4.18)$$

We also have (cf. (4.13), (4.14))

$$\frac{\partial \tilde{v}}{\partial t}(x,y,t) = 0, \quad \frac{\partial \tilde{w}}{\partial \nu}(x,y,t) = 0, \quad (x,y) \in B_1, \quad t \in [0,T] . \qquad (4.19)$$

The boundary values of $\tilde{v}$ and $\tilde{w}$ are therefore _overspecified_ on $B_1 \times [0,T]$. The proof

of approximate controllability, where it can be carried through, depends upon being able to

use this overspecification to show that

$$\tilde{v}(x,y,t) \equiv 0, \quad \tilde{w}(x,y,t) \equiv 0, \quad (x,y) \in R, \quad t \in [0,T] ,$$

and therefore to conclude that the implication (4.11) is indeed valid so that $R(U,T)$ is

dense in $\hat{H}_1$ and hence in H. We carry this argument out for the case in which R is a

rectangle and B_1 is one of its sides in Section 5.

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Following the development in [6], it may be seen that our system is exactly controllable in $\hat{H}_1$, using the control space $U = L^2(B_1 \times [0,T])$, just in case

$$\|\alpha \frac{\partial \tilde{v}}{\partial \nu} + \beta \frac{\partial \tilde{w}}{\partial t}\|_{L^2(B_1 \times [0,T])} \geq K\|(\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)\|_{\hat{H}} \qquad (4.20)$$

for some $K > 0$. In general this is a very difficult result to obtain but we are able to obtain exact controllability, by other means, for the case where R is a disc in R^2 and $B_1 = B$ is its boundary, a circle. This result is developed in _nction 6 where it will be seen that it is heavily dependent on certain properties of the Bessel functions.

5. THE CASE R = A RECTANGLE, B_1 = ONE SIDE.

The work here can be carried out for a rectangle with arbitrary dimensions, but all essential ideas are contained in the notationally simpler case

$$R = \{(x,y) | 0 \leqslant x \leqslant \pi, \ 0 \leqslant y \leqslant \pi\}$$

to which attention is restricted henceforth. We will assume that B_1, the portion of the boundary on which control is exercised, is one side of R, without loss of generality it is the set

$$B_1 = \{(\pi,y) | 0 \leqslant y \leqslant \pi\} \ . \tag{5.1}$$

We consider then $\tilde{v}, \tilde{w}$ satisfying (4.1), (4.2) in $R \times [0,T]$ for some $T > 0$, and also satisfying boundary conditions

$$\frac{\partial \tilde{v}}{\partial t} (x,y,t) = 0, \ \frac{\partial \tilde{w}}{\partial \nu} (x,y,t) = 0, \ (x,y) \in B = \partial R, \ t \in [0,T] \ , \tag{5.2}$$

$$\alpha \frac{\partial \tilde{v}}{\partial \nu} (\pi,y,t) + \beta \frac{\partial \tilde{w}}{\partial t} (\pi,y,t)$$

$$= \alpha \frac{\partial \tilde{v}}{\partial x} (\pi,y,t) + \beta \frac{\partial \tilde{w}}{\partial t} (\pi,y,t) = 0, \ \ 0 \leqslant y \leqslant \pi, \ \ t \in [0,T] \ . \tag{5.3}$$

We may assume without loss of generality, since the wave equation is time reversible with either Dirichlet or Neumann boundary conditions, that $\tilde{v}$ and $\tilde{w}$ are extended to satisfy (4.1), (4.2) on $-\infty < t < \infty$ and that the boundary conditions (5.2) hold for $(x,y) \in B$, $t \in (-\infty,\infty)$. We may not assume that the boundary condition (5.3) is applicable beyond $[0,T]$, however, if controls are restricted to have support in $B_1 \times [0,T]$. Let $\delta > 0$ and let $s(t)$ be an arbitrary function in $C^\infty(-\infty,\infty)$ with support in $(-\delta,\delta)$. Define

$$\hat{v}(x,y,t) = \int_{-\infty}^{\infty} s(t - \tau)\tilde{v}(x,y,\tau)d\tau \ , \tag{5.4}$$

$$\hat{w}(x,y,t) = \int_{-\infty}^{\infty} s(t - \tau)\tilde{w}(x,y,\tau)d\tau \ . \tag{5.5}$$

Then $\hat{v}, \hat{w}$ are solutions of the wave equations (4.1), (4.2) satisfying boundary conditions

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$$\frac{\partial \hat{v}}{\partial t} (x,y,t) = 0, \quad \frac{\partial \hat{w}}{\partial \nu} (x,y,t) = 0, \quad (x,y) \in B = \partial R, \quad -\infty < t < \infty, \qquad (5.6)$$

while

$$\alpha \frac{\partial \hat{v}}{\partial x} (\pi,y,t) + \beta \frac{\partial \hat{w}}{\partial t} (\pi,y,t) = 0, \quad 0 < y < \pi, \quad t \in [\delta, T - \delta] . \qquad (5.7)$$

Moreover, it can be shown that $\hat{v}, \hat{w}$ are of class C^∞ for $(x,y) \in R$, $-\infty < t < \infty$. If we can show $\hat{v} \equiv 0$, $\hat{w} \equiv 0$ for any such choice of s, then $\tilde{v} \equiv 0$, $\tilde{w} \equiv 0$.

Let us define, for $(x,y) \in R$, $-\infty < t < \infty$,

$$\phi(x,y,t) = \alpha \frac{\partial \hat{v}}{\partial x} (x,y,t) + \beta \frac{\partial \hat{w}}{\partial t} (x,y,t) . \qquad (5.8)$$

From (5.7) we have

$$\phi(\pi,y,t) = 0, \quad 0 < y < \pi, \quad t \in [\delta, T - \delta] . \qquad (5.9)$$

Since α and β are constants we have

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}, \quad (x,y) \in R, \quad -\infty < t < \infty . \qquad (5.10)$$

Let us note that, since $\hat{v}$ satisfies the wave equation in $R \cup B$,

$$\alpha \frac{\partial^2 \hat{v}}{\partial t^2} (x,y,t) + \beta \frac{\partial^2 \hat{w}}{\partial t \partial x} (x,y,t)$$

$$= \alpha \Big[\frac{\partial^2 \hat{v}}{\partial x^2} (x,y,t) + \frac{\partial^2 \hat{v}}{\partial y^2} (x,y,t) \Big] + \beta \frac{\partial^2 \hat{w}}{\partial t \partial x} (x,y,t) . \qquad (5.11)$$

Setting $x = \pi$ in (5.11) and differentiating the identities in (5.6) with respect to t, we see that the left hand side vanishes. Then, comparing (5.11) with (5.8)

$$\frac{\partial \phi}{\partial x} (\pi,y,t) = -\alpha \frac{\partial^2 \hat{v}}{\partial y^2} (\pi,y,t) \equiv a(y), \quad 0 < y < \pi, \quad \delta < t < T - \delta , \qquad (5.12)$$

the last identity being valid as a consequence of the first condition in (5.6).

The two conditions, (5.8) and (5.12), satisfied by ϕ at the boundary $x = \pi$ enable us to use Holmgren's uniqueness theorem (see [5] or [13], e.g.) in much the same way as it

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was used in the proof of the approximate controllability of the wave equation in [22], [23] to see that if

$$T > 2 + 2\delta \tag{5.13}$$

then ϕ must be independent of t for $1 + \delta < t < T - 1 - \delta$, i.e.

$$\phi(x,y,t) = \phi(x,y), \quad (x,y) \in R, \quad 1 + \delta < t < T - 1 - \delta. \tag{5.14}$$

Because $\hat{v}$ and $\hat{w}$ satisfy the wave equation in R with the homogeneous boundary conditions (5.6), and are of class C^∞ in $R \cup B$, we have C^∞- convergent expansions

$$\hat{v}(x,y,t) = \hat{v}_0(x,y) + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (v_{kj}e^{i\omega_{kj}t} + \bar{v}_{kj}e^{-\omega_{kj}t})\sin kx \sin jy, \tag{5.15}$$

$$\hat{w}(x,y,t) = \hat{w}_0 + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (w_{kj}e^{i\omega_{kj}t} + \bar{w}_{kj}e^{-i\omega_{kj}t})\cos kx \cos jy, \tag{5.16}$$

where

$$w_{kj} = \sqrt{k^2 + j^2}, \tag{5.17}$$

$\hat{v}_0(x,y)$ is a C^∞ function in $R \cup B$ such that (cf. (4.10))

$$\hat{v}_0(x,y) = 0, \quad (x,y) \in B - \{(\pi,y)|0 < y < \pi\} \tag{5.18}$$

and $\hat{w}_0$ is a constant. Then, from (5.8),

$$\phi(x,y,t) - \alpha \frac{\partial \hat{v}_0(x,y)}{\partial x} =$$

$$= \sum_{k=1}^{\infty} \cos kx \Big[\sum_{j=1}^{\infty} (\alpha k v_{kj}\sin jy + i\beta\omega_{kj}w_{kj}\cos jy)e^{i\omega_{kj}t}$$

$$+ \sum_{j=1}^{\infty} (\alpha k \bar{v}_{kj}\sin jy - i\beta\omega_{kj}\bar{w}_{kj}\cos jy)e^{-i\omega_{kj}t} \Big], \tag{5.19}$$

still C^∞- convergent for $(x,y) \in R \cup B$, $-\infty < t < \infty$. Noting (5.14), we see that the left hand side takes the form

$$\phi(x,y,t) - \alpha \frac{\partial \hat{v}_0(x,y)}{\partial x} = \phi(x,y) - \alpha \frac{\partial \hat{v}_0(x,y)}{\partial x} \equiv \hat{\phi}(x,y),$$

$$1 + \delta < t < T - 1 - \delta. \tag{5.20}$$

We now strengthen (5.13) to

$$T > 4 + 2\delta \qquad (5.21)$$

and we see that the time interval in (5.14), (5.20) has length > 2, i.e.

$$T - 1 - \delta - (1 + \delta) = T - (2 + 2\delta) > 2 . \qquad (5.17)$$

Since the functions $\sqrt{\frac{2}{\pi}} \cos kx$ are orthonormal on $0 < x < \pi$, we conclude from (5.19), (5.20) that for $k = 1,2,3,\ldots$

$$\sum_{j=1}^{\infty} (\alpha k v_{kj} \sin jy + i\beta\omega_{kj} w_{kj} \cos jy) e^{i\omega_{kj}t}$$

$$+ \sum_{j=1}^{\infty} (\alpha k \bar{v}_{kj} \sin jy - i\beta\omega_{kj} \bar{w}_{kj} \cos jy) e^{-i\omega_{kj}t}$$

$$= \frac{2}{\pi} \int_0^{\pi} \hat{\phi}(x,y) \cos kx \, dx \equiv \hat{\phi}_k(y), \quad 1 + \delta < t < T - 1 - \delta . \qquad (5.22)$$

Classical results of Levinson and Schwartz ([17], [27]), which have frequently been used in control studies of this type (see, e.g., [12], [21]), can now be used to show that for each fixed k, the exponential functions

$$e^{\pm i\omega_{kj}t} = e^{\pm i\sqrt{k^2+j^2}\,t}, \qquad j = 1,2,3,\ldots ,$$

together with the constant function 1 are strongly independent in $L^2(I)$ for any t-interval I of length > 2. This clearly contradicts (5.22) unless we have

$$\hat{\phi}_k(y) \equiv 0, \quad 0 < y < \pi \qquad (5.23)$$

and

$$\alpha k v_{kj} \sin jy + i\beta\omega_{kj} w_{kj} \cos jy = 0, \quad 0 < y < \pi, \quad j = 1,2,3,\ldots .$$

But then, since for each j $\sin jy$ and $\cos jy$ are independent on $0 < y < \pi$ and since none of α, k, β, ω_{kj} are zero, we conclude that

$$v_{kj} = 0, \quad w_{kj} = 0, \quad k = 1,2,3,\ldots, \quad j = 1,2,3,\ldots . \qquad (5.24)$$

Since (5.22), (5.23) show that

$$\hat{\phi}(x,y) = \sum_{k=1}^{\infty} \hat{\phi}_k(y) \cos kx = 0 ,$$

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(5.19) gives

$$\phi(x,y,t) = \phi(x,y) = \alpha \frac{\partial \hat{v}_0(x,y)}{\partial x}, \quad (x,y) \in R, \qquad (5.25)$$

$$1 + \delta < t < T - 1 - \delta .$$

Noting (5.15) and (5.16) and the fact that $\hat{v}(0,y,t) \equiv 0$, we conclude from (5.23) that

$$\left. \begin{array}{l} \hat{v}(x,y,t) \equiv \hat{v}_0(x,y) , \\[2mm] \hat{w}(x,y,t) \equiv \hat{w}_0 , \end{array} \right\} \quad 1 + \delta < t < T - 1 - \delta . \qquad (5.26)$$

Since $v(x,y,t) \equiv v_0(x,y)$ is a solution of the wave equation with (cf. (5.18))

$$\hat{v}_0(x,y) = 0, \quad (x,y) \in B - \{(\pi,y) | 0 < y < \pi\}$$

it must in fact be a solution of Laplace's equation there. Then we compute

$$\int_R \left[\left(\frac{\partial \hat{v}_0}{\partial x}(x,y) \right)^2 + \left(\frac{\partial \hat{v}_0}{\partial y}(x,y) \right)^2 + \hat{v}_0(x,y) \left(\frac{\partial^2 \hat{v}_0}{\partial x^2}(x,y) + \frac{\partial^2 v_0}{\partial y^2}(x,y) \right) \right] dxdy$$

$$= \int_R \operatorname{div}(\hat{v}_0(x,y) \operatorname{grad} \hat{v}_0(x,y)) dxdy$$

$$= \int_B \hat{v}_0(x,y) \operatorname{grad} \hat{v}_0(x,y) \cdot \nu(x,y) ds = \int_0^\pi \hat{v}_0(\pi,y) \frac{\partial \hat{v}_0}{\partial x}(\pi,y) dy . \qquad (5.27)$$

Combining (5.9) and (5.25) with the fact that $\hat{v}_0$ satisfies Laplace's equation we conclude from (5.27) that

$$\int_R \left[\left(\frac{\partial \hat{v}_0}{\partial x}(x,y) \right)^2 + \left(\frac{\partial \hat{v}_0}{\partial y}(x,y) \right)^2 \right] dxdy = 0$$

and this, together with (5.18), implies

$$\hat{v}_0(x,y) \equiv 0 . \qquad (5.28)$$

Combining (5.26) and (5.28) we conclude that

$$\left. \begin{array}{l} \hat{v}(x,y,t) \equiv 0 \\[2mm] \hat{w}(x,y,t) \equiv \hat{w}_0 \end{array} \right\} \quad (x,y) \in R, \quad -\infty < t < \infty \qquad (5.29)$$

the result for $-\infty < t < \infty$ being an immediate consequence of the result for

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$1 + \delta \leqslant t \leqslant T - 1 - \delta$. Since this is true for every $\delta > 0$ and every $s(t)$ in (5.4), (5.5), we conclude that a comparable result obtains for $\hat{v}, \hat{w}$ in (4.11), (5.2), (5.3). It follows (since $w = $ constant is a zero state in $\hat{H}$ and in H) that (cf. (4.9) ff.)

$$\|(\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)\|_{\hat{H}} = \|(\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)\|_{H} = 0$$

and, from the discussion in Section 4, the approximate controllability result follows.

6. SOME EXACT CONTROLLABILITY RESULTS IN THE CASE OF A CIRCULAR CYLINDER

We consider now the case $\Omega = R \times (-\infty,\infty)$ with

$$R = \{(x,y) | x^2 + y^2 < 1\} ,$$

$$B = \partial R = \{(x,y) | x^2 + y^2 = 1\} .$$

With introduction of the usual polar coordinates r,θ, the equations (4.1), (4.2) now become

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \tag{6.1}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \tag{6.2}$$

and the boundary conditions (4.3), (4.4) are transformed to

$$\frac{\partial v}{\partial t} (1,\theta,t) = \alpha u(\theta,t) , \tag{6.3}$$

$$\frac{\partial w}{\partial r} (1,\theta,t) = \beta u(\theta,t) . \tag{6.4}$$

Writing

$$v(r,\theta,t) = \sum_{k=-\infty}^{\infty} v_k(r,t) e^{ik\theta}, \quad v_{-k} = \bar{v}_k , \tag{6.5}$$

$$w(r,\theta,t) = \sum_{k=-\infty}^{\infty} w_k(r,t) e^{ik\theta}, \quad w_{-k} = \bar{w}_k , \tag{6.6}$$

$$u(\theta,t) = \sum_{k=-\infty}^{\infty} u_k(t) e^{ik\theta} \tag{6.7}$$

we arrive at an infinite collection of control problems in the single space dimension, r:

$$\frac{\partial^2 v_k}{\partial t^2} = \frac{\partial^2 v_k}{\partial r^2} + \frac{1}{r} \frac{\partial v_k}{\partial r} - \frac{k^2}{r^2} v_k = 0, \quad -\infty < k < \infty , \tag{6.8}$$

$$\frac{\partial^2 w_k}{\partial t^2} = \frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r} \frac{\partial w_k}{\partial r} - \frac{k^2}{r^2} w_k = 0, \quad -\infty < k < \infty , \tag{6.9}$$

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$$\frac{\partial v_k}{\partial t}(1,t) = \alpha u_k(t), \quad -\infty < k < \infty, \tag{6.10}$$

$$\frac{\partial w_k}{\partial r}(1,t) = \beta u_k(t), \quad -\infty < k < \infty. \tag{6.11}$$

We will first treat the equation (4.1) with the boundary condition (4.3) which, as we have seen, reduces to the set of problems (6.8), (6.10), $-\infty < k < \infty$. With

$$z(r,\theta,t) = \sum_{k=-\infty}^{\infty} z_k(r,t)e^{ik\theta} = \sum_{k=-\infty}^{\infty} \frac{\partial v_k(r,t)}{\partial t} e^{ik\theta} = \frac{\partial v}{\partial t}(r,\theta,t)$$

we have the equivalent first order systems

$$\frac{\partial}{\partial t}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ L_{|k|} & 0 \end{pmatrix}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} = L_{|k|}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} \tag{6.12}$$

where $L_{|k|}$ is the differential operator on the right hand side of (6.8). The boundary conditions (6.10) become

$$z_k(1,t) = \alpha u_k(t), \quad -\infty < k < \infty. \tag{6.13}$$

The eigenvalues of the operator $L_{|k|}$ with the corresponding homogeneous boundary condition

$$z_k(1,t) = 0 \tag{6.14}$$

are

$$0, \quad \pm i\omega_{|k|,\ell}, \quad \ell = 1,2,3,\dots,$$

where $\omega_{|k|,\ell}$ is the ℓ-th positive zero of the Bessel function $J_{|k|}(r)$ of order $|k|$. The corresponding vector eigenfunctions are

$$\begin{pmatrix} \phi_{|k|0}(r) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \phi_{|k|\ell}(r) \\ \pm i\omega_{|k|,\ell}\,\phi_{|k|\ell}(r) \end{pmatrix}, \quad \begin{array}{l} -\infty < k < \infty \\ \ell = 1,2,3,\dots, \end{array}$$

where

$$\phi_{|k|,0}(r) = A_{|k|,0}r^{|k|}, \quad -\infty < k < \infty, \tag{6.15}$$

$$\phi_{|k|\ell}(r) = A_{|k|\ell} J_{|k|}(\omega_{|k|,\ell} r), \qquad \begin{array}{l} -\infty < k < \infty, \\ \ell = 1,2,3,\ldots \end{array}$$

The normalization coefficients $A_{|k|,0}$, $A_{|k|,\ell}$ are chosen so that

$$\int_0^1 r|\phi_{|k|,0}(r)|^2 dr = \frac{1}{2\pi}, \quad \int_0^1 r|\phi_{|k|,\ell}(r)|^2 dr = \frac{1}{2\pi}, \quad \ell = 1,2,3,\ldots . \qquad (6.16)$$

Thus

$$A_{|k|,0} = \sqrt{\frac{|k| + 1}{\pi}}, \qquad -\infty < k < \infty, \qquad (6.17)$$

while, as may be seen from [5], e.g.

$$A_{|k|,\ell} = \frac{\omega_{|k|,\ell}}{\sqrt{\pi} \, J'_{|k|,\ell}(\omega_{|k|,\ell})} \qquad (6.18)$$

The state space in which we wish to work, for the present at least, is (cf. (2.18))

$$\tilde{H} = \left\{ \begin{pmatrix} v \\ z \end{pmatrix} \middle| v \in H^1(\mathbb{R}), \ z \in L^2(\mathbb{R}) \right\}$$

with the inner product

$$\left(\begin{pmatrix} v_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ z_2 \end{pmatrix} \right) = \int_{\mathbb{R}} (\nabla v_1 \cdot \overline{\nabla v_2} + z_1 \overline{z_2}) dx dy$$

and associated norm. Since the $\phi_{|k|,\ell}$ satisfy the homogeneous boundary condition (6.14) one easily sees that

$$\left\| \begin{pmatrix} \phi_{|k|,0} e^{ik\theta} \\ 0 \end{pmatrix} \right\|_{\tilde{H}}^2 = - \int_{\mathbb{R}} \phi_{|k|,0} e^{ik\theta} \overline{\Delta(\phi_{|k|,0} e^{ik\theta})} dx dy$$

$$+ \int_{\partial \mathbb{R}} \phi_{|k|,0} e^{ik\theta} \frac{\overline{\partial \phi_{|k|,0}}}{\partial r} e^{-ik\theta} d\theta + \frac{|k| + 1}{\pi} \int_0^{2\pi} |k| d\theta$$

$$= 2|k|(|k| + 1), \quad -\infty < k < \infty, \qquad (6.19)$$

while

$$\left| \begin{pmatrix} \phi_{|k|,\ell} \\ \pm i\omega_{|k|,\ell}\phi_{|k|,\ell} \end{pmatrix} \right|_{\widetilde{H}}^2 = \lambda_{|k|,\ell} \int_R |\varepsilon_{|k|,\ell}|^2 \, dxdy$$

$$+ \int_R \nabla\phi_{|k|,\ell} \cdot \overline{\nabla\phi_{|k|,\ell}} \, dxdy = 2\lambda_{|k|,\ell} \int_R |\phi_{|k|,\ell}|^2 dxdy = 2\lambda_{|k|,\ell} \quad (6.20)$$

where

$$\lambda_{|k|,\ell} = (\omega_{|k|,\ell})^2, \quad -\infty < k < \infty, \quad \ell = 1,2,3,\dots \ .$$

The state $\begin{pmatrix} \phi_{0,0} \\ 0 \end{pmatrix}$ has zero norm in $\widetilde{H}$. Nevertheless we will not neglect this component.

If v, $\widetilde{v}$ both satisfy the wave equation and (6.3), (4.13) on ∂R with initial state (4.5) for v we have (cf. (4.16))

$$\left(\begin{pmatrix} v(\cdot,\cdot,T) \\ z(\cdot,\cdot,T) \end{pmatrix}, \begin{pmatrix} \widetilde{v}(\cdot,\cdot,T) \\ z(\cdot,\cdot,T) \end{pmatrix} \right)_{\widetilde{H}} = \alpha \int_0^T \int_{B=\partial R} u(x,y,t) \, \overline{\frac{\partial\widetilde{v}}{\partial\nu}(x,y,t)} \, dsdt \ . \quad (6.21)$$

It may be shown that this result is valid for all u for which the solution (in the generalized sense) v lies in $\widetilde{H}$ and varies continuously with respect to t. This class of controls u is discussed in [19] and is known to include, e.g., $u \in C([0,T];H^{1/2}(B))$. If we assume $\begin{pmatrix} v \\ z \end{pmatrix}$ given by the $\widetilde{H}$- convergent series

$$\begin{pmatrix} v(\cdot,\cdot,t) \\ z(\cdot,\cdot,t) \end{pmatrix} = \sum_{k=-\infty}^{\infty} v_{k,0}(t)(\phi_{|k|,0}e^{ik\theta}) +$$

$$\sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \left[v_{k,\ell}^+(t) \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix} + v_{k,\ell}^-(t) \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ -i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix} \right]$$

and successively let

$$\begin{pmatrix} \widetilde{v}(\cdot,\cdot,t) \\ \widetilde{z}(\cdot,\cdot,t) \end{pmatrix} = \begin{pmatrix} \phi_{|k|,0}e^{ik\theta} \\ 0 \end{pmatrix}, \quad e^{i\omega_{|k|,\ell}(t-T)} \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix},$$

$$e^{-i\omega_{|k|,\ell}(t-T)} \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ -i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix}, \quad \begin{array}{l} -\infty < k < \infty \\ \ell = 1,2,3,\dots \end{array} \quad (6.22)$$

for $T > 0$ we arrive at the equations

$$2|k|(|k| + 1)v_{k,0}(T) = \alpha \int_0^T \int_0^{2\pi} u(\theta,t) \overline{\frac{\partial \phi_{|k|,0}}{\partial r}(1)} e^{-ik\theta} d\theta dt$$

$$= 2\pi\alpha \overline{\frac{\partial \phi_{|k|,0}}{\partial r}(1)} \int_0^T u_k(t)dt , \qquad (6.23)$$

$$2\lambda_{|k|,\ell}v^+_{k,\ell}(T) = \alpha \int_0^T \int_0^{2\pi} u(\theta,t) e^{i\omega_{|k|,\ell}(T-t)} \overline{\frac{\partial \phi_{|k|,\ell}}{\partial r}(1)} e^{-ik\theta} d\theta dt$$

$$= 2\pi\alpha \overline{\frac{\partial \phi_{|k|,\ell}}{\partial r}(1)} \int_0^T e^{i\omega_{|k|,\ell}(T-t)} u_k(t)dt , \qquad (6.24)$$

$$2\lambda_{|k|,\ell}v^-_{k,\ell}(T) = \alpha \int_0^T \int_0^{2\pi} u(\theta,t) e^{-i\omega_{|k|,\ell}(T-t)} \overline{\frac{\partial \phi_{|k|,\ell}}{\partial r}(1)} e^{-ik\theta} d\theta dt$$

$$= 2\pi\alpha \overline{\frac{\partial \phi_{|k|,\ell}}{\partial r}(1)} \int_0^T e^{-i\omega_{|k|,\ell}(T-t)} u_k(t)dt . \qquad (6.25)$$

Thus the Dirichlet boundary control problem for (6.8), (6.10) is reduced to a moment problem (6.23), (6.24), (6.25) for which $u_k(t)$ must be a solution. We proceed in much the same way with the Neumann boundary control problem for (6.9), (6.11). We let

$$\zeta(r,\theta,t) = \sum_{k=-\infty}^{\infty} \zeta_k(r,t)e^{ik\theta} = \sum_{k=-\infty}^{\infty} \frac{\partial w_k(r,t)}{\partial t} e^{ik\theta} = \frac{\partial w}{\partial t}(r,\theta,t)$$

and obtain, in place of (6.12),

$$\frac{\partial}{\partial t} \begin{pmatrix} w_k(r,t) \\ \zeta_k(r,t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ M_{|k|} & 0 \end{pmatrix} \begin{pmatrix} w_k(r,t) \\ \zeta_k(r,t) \end{pmatrix} = M_{|k|} \begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} . \qquad (6.26)$$

The boundary conditions are now

$$\frac{\partial w_k}{\partial r}(1,t) = \beta u_k(t), \qquad -\infty < k < \infty .$$

The eigenvalues of $M_{|k|}$ with the corresponding homogeneous boundary condition

$$\frac{\partial w_k}{\partial r}(1,t) = 0$$

are, for $k = 0$,

$$0, \quad \pm i\nu_{0,\ell}, \quad \ell = 1,2,3,\ldots \, ,$$

where $\nu_{0,\ell}$ is the ℓ-th zero of the differentiated Bessel function, $j_0'(r)$, of order 0, and, for $k \neq 0$,

$$\pm i\nu_{|k|,\ell}, \quad \ell = 1,2,3,\ldots \, ,$$

where $\nu_{k,\ell}$ is the ℓ-th zero of $J_k'(r)$. In the case $k = 0$ the eigenvalue 0 has double multiplicity. The special solutions taking the place of (6.22) in this case are

$$\begin{pmatrix} \widetilde{w}(\cdot,\cdot,t) \\ \widetilde{\zeta}(\cdot,\cdot,t) \end{pmatrix} = \begin{pmatrix} \psi_{00} \\ 0 \end{pmatrix} , \quad \begin{pmatrix} (t - T)\psi_{00} \\ \psi_{00} \end{pmatrix} \tag{6.27}$$

where ψ_{00} is such that (cf. (6.16))

$$\int_0^1 r\psi_{00}^2 \, dr = \frac{1}{2\pi}, \quad \text{i.e. } \psi_{00} = \frac{1}{\sqrt{\pi}} \, .$$

In all of the other cases the vector eigenfunctions take the form

$$\begin{pmatrix} \psi_{|k|,\ell}(r) \\ \pm i\nu_{|k|,\ell}\psi_{|k|,\ell}(r) \end{pmatrix}, \quad -\infty < k < \infty, \quad \ell = 1,2,3,\ldots$$

where

$$\psi_{|k|,\ell}(r) = B_{|k|,\ell}J_{|k|}(\nu_{|k|,\ell}r), \quad \begin{array}{l} -\infty < k < \infty, \\ \ell = 1,2,3,\ldots \, , \end{array}$$

the normalization coefficients

$$B_{|k|,\ell} = \frac{\nu_{|k|,\ell}}{\sqrt{\pi} \, (u_{|k|,\ell} - k^2)^{1/2} J_{|k|}(\nu_{|k|,\ell})} \tag{6.28}$$

selected so that

$$\int_0^R r|\psi_{|k|,\ell}(r)|^2 dr = \frac{1}{2\pi} \, .$$

The corresponding special solutions of the homogeneous equation are

$$\begin{pmatrix} \widetilde{w}(\cdot,\cdot,t) \\ \widetilde{\zeta}(\cdot,\cdot,t) \end{pmatrix} = e^{i\nu_{|k|,\ell}(t-T)} \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} ,$$

$$e^{-i\nu_{|k|,\ell}(t-T)} \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ -i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} . \qquad (6.29)$$

As in (6.20) it may be seen that

$$\left\| \begin{pmatrix} \psi_{|k|,\ell} \\ \pm i\nu_{|k|,\ell}\psi_{|k|,\ell} \end{pmatrix} \right\|^2_{\widetilde{H}} = 2\mu_{|k|,\ell} , \quad \mu_{|k|,\ell} = (\nu_{|k|,\ell})^2 .$$

Let w satisfy the wave equation and (6.4) with $w(x,y,0) \equiv 0$, $\zeta(x,y,0) = \frac{\partial w}{\partial t}(x,y,0) \equiv 0$ in R. We expand $\begin{pmatrix} w \\ \zeta \end{pmatrix}$ in the form

$$\begin{pmatrix} w(\cdot,\cdot,t) \\ \zeta(\cdot,\cdot,t) \end{pmatrix} = w_{00}(t) \begin{pmatrix} \psi_{00} \\ 0 \end{pmatrix} + \zeta_{00}(t) \begin{pmatrix} 0 \\ \psi_{00} \end{pmatrix}$$

$$+ \sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \left[w_{k,\ell}^{+}(t) \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} + w_{k,\ell}^{-}(t) \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ -i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} \right]$$

If $\widetilde{w}$ satisfies the wave equation and the homogeneous boundary condition (cf. (4.14))
$$\frac{\partial \widetilde{w}}{\partial\nu}(x,y,t) = 0, \quad (x,y) \in B, \quad t > 0 ,$$
we find (cf. (4.16), (6.21)) that

$$\left(\begin{pmatrix} w(\cdot,\cdot,T) \\ \zeta(\cdot,\cdot,T) \end{pmatrix}, \begin{pmatrix} \widetilde{w}(\cdot,\cdot,T) \\ \widetilde{\zeta}(\cdot,\cdot,T) \end{pmatrix} \right)_{\widetilde{H}} = \beta \int_0^T \int_{B=\partial R} u(x,y,t) \overline{\frac{\partial \widetilde{w}}{\partial t}(x,y,t)} \, ds dt . \qquad (6.30)$$

Employing (6.29), (6.3) successively for $\begin{pmatrix} \widetilde{w} \\ \widetilde{\zeta} \end{pmatrix}$ we arrive at the equations, for $-\infty < k < \infty$, $\ell = 1,2,3,\ldots$,

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$$2\mu_{|k|,\ell} w_{k,\ell}^{+}(T) = \beta \int_0^T \int_0^{2\pi} u(\theta,t) i\nu_{|k|,\ell} e^{i\nu_{|k|,\ell}(T-t)} \psi_{|k|,\ell}(1) e^{-ik\theta} d\theta dt$$

$$= 2\pi\beta i\nu_{|k|,\ell} \overline{\psi_{|k|,\ell}(1)} \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t) dt , \qquad (6.31)$$

$$2\mu_{|k|,\ell} w_{k,\ell}^{-}(T) = -\beta \int_0^T \int_0^{2\pi} u(\theta,t) i\nu_{|k|,\ell} e^{-i\nu_{|k|,\ell}(T-t)} \psi_{|k|,\ell}(1) e^{-ik\theta} d\theta dt$$

$$= -2\pi\beta i\nu_{|k|,\ell} \overline{\psi_{|k|,\ell}(1)} \int_0^T e^{-i\nu_{|k|,\ell}(T-t)} u_k(t) dt . \qquad (6.32)$$

We find also, taking $\binom{\tilde{w}}{\zeta}$ in the second form given in (6.27), that

$$\zeta_{00}(T) = \beta \int_0^T \int_0^{2\pi} u(\theta,t) \overline{\psi_{00}} d\theta dt = 2\pi\beta \overline{\psi_{00}} \int_0^T u_0(t) dt . \qquad (6.33)$$

Since this must be true for all T and $\dfrac{d}{dt} w_{00}(t) = \zeta_{00}(t)$, we have also

$$w_{00}(T) = 2\pi\beta \overline{\psi_{00}} \int_0^T (T - t) u_0(t) dt \qquad (6.34)$$

Since $\mu_{|k|,\ell} = (\nu_{|k|,\ell})^2$, (6.31), (6.32) become

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w_{k,\ell}^{+}(T) = \overline{\psi_{|k|,\ell}(1)} \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t) dt$$

$$= B_{|k|,\ell} J_{|k|,\ell}(\nu_{|k|,\ell}) \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t) dt \qquad (6.35)$$

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w_{k,\ell}^{-}(T) = -B_{|k|,\ell} J_{|k|,\ell}(\nu_{|k|,\ell}) \int_0^T e^{-i\nu_{|k|,\ell}(T-t)} u_k(t) dt . \qquad (6.36$$

Taking account of the fact that

$$\overline{\frac{\partial \phi_{|k|,\ell}}{\partial r}(1)} = \omega_{|k|,\ell} A_{|k|,\ell} \frac{\partial J_{|k|,\ell}}{\partial r}(\omega_{|k|,\ell})$$

(6.24) and (6.25) yield

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$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^+_{k,\ell}(T) = A_{|k|,\ell} \frac{\partial J_{|k|,\ell}}{\partial r}(\omega_{|k|,\ell}) \int_0^T e^{i\omega_{|k|,\ell}(T-t)} u_k(t)dt , \qquad (6.37)$$

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^-_{k,\ell}(T) = A_{|k|,\ell} \frac{\partial J_{|k|,\ell}}{\partial r}(\omega_{|k|,\ell}) \int_0^T e^{-i\omega_{|k|,\ell}(T-t)} u_k(t)dt . \qquad (6.38)$$

On the other hand

$$\frac{\partial \phi_{|k|,0}}{\partial r}(1) = A_{|k|,0}|k|$$

so (6.23) gives

$$\frac{|k| + 1}{\pi\alpha} v_{k,0}(T) = A_{|k|,0} \int_0^T u_k(t)dt . \qquad (6.39)$$

Using the formula (6.18) and (6.28) for $A_{|k|,\ell}$ and $B_{|k|,\ell}$ we have

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w^+_{k,\ell}(T) = \frac{\nu_{|k|,\ell}}{\sqrt{\pi} (\mu_{|k|,\ell} - k^2)^{1/2}} \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t)dt \qquad (6.40)$$

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w^-_{k,\ell}(T) = \frac{-\nu_{|k|,\ell}}{\sqrt{\pi} (\mu_{|k|,\ell} - k^2)^{1/2}} \int_0^T e^{-i\nu_{|k|,\ell}(T-t)} u_k(t)dt \qquad (6.41)$$

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^+_{k,\ell}(T) = \frac{1}{\sqrt{\pi}} \int_0^T e^{i\omega_{|k|,\ell}(T-t)} u_k(t)dt \qquad (6.42)$$

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^-_{k,\ell}(T) = \frac{1}{\sqrt{\pi}} \int_0^T e^{-i\omega_{|k|,\ell}(T-t)} u_k(t)dt \qquad (6.43)$$

The equations (6.39) become, in view of (6.17),

$$\frac{\sqrt{2}\ \sqrt{|k|(|k| + 1)}}{\pi\alpha} v_{k,0}(T) = \frac{\sqrt{2|k|}}{\sqrt{\pi}} \int_0^T u_k(t)dt . \qquad (6.44)$$

This is valid, but meaningless, for $k = 0$. It is easy to see that in the case $k = 0$ we

should use

$$\frac{1}{\sqrt{\pi}\ \alpha}\ v_{00}(T) = \int_0^T u_k(t)dt \ . \tag{6.45}$$

The equations (6.33) and (6.34) are left as they appear. We note that all of the
coefficients

$$\frac{v_{|k|,\ell}}{\sqrt{\pi}\ (\mu_{|k|,\ell} - k^2)^{1/2}}, \quad \frac{1}{\sqrt{\pi}}, \quad \frac{\sqrt{2|k|}}{\sqrt{\pi}}, \quad k \neq 0, \quad 2\pi\beta \tag{6.45}$$

are bounded away from zero, uniformly with respect to k.

It is also possible to show, using the work [10], [11] of K. D. Graham, that the
numbers

$$0, \ v_{|k|,1}, \ \omega_{|k|,1}, \ v_{|k|,2}, \ \omega_{|k|,2}, \cdots, v_{|k|,j}, \ \omega_{|k|,j}, \cdots$$

are separated by a gap at least equal to $\pi/2$ again uniformly with respect to k.
Applying the result [14] of A. E. Ingham along with the work of Duffin and Schaeffer [7],
much as in [12], [2], [3], we conclude the existence of functions $u_k(t)$ in $L^2[0,T]$, for
any fixed $T > 4$, solving the above moment problems, $-\infty < k < \infty$. Moreover, the result of
Ingham implies as explained in [12], [26], that for each k

$$c^{-2}N_k^2 < \int_0^T |u_k(t)|^2 dt < c^2 N_k^2$$

where

$$N_k^2 = 2|k|(|k| + 1)|v_{k,0}(T)|^2$$

$$+ \sum_{\ell=1}^{\infty} \lambda_{|k|,\ell}|v_{k,\ell}^+(T)|^2 + \sum_{\ell=1}^{\infty} \lambda_{|k|,\ell}|v_{k,\ell}^-(T)|^2$$

$$+ \sum_{\ell=1}^{\infty} \mu_{|k|,\ell}|w_{k,\ell}^+(T)|^2 + \sum_{\ell=1}^{\infty} \mu_{|k|,\ell}|w_{k,\ell}^-(T)|^2$$

$k = \pm 1, \pm 2, \ldots$. For $k = 0$ we must add $|\zeta_{00}(T)|^2 + |w_{00}(T)|^2$. Since

$$\int_0^T \int_0^{2\pi} |u(\theta,t)|^2 d\theta dt = \sum_{k=-\infty}^{\infty} \int_0^T |u_k(t)|^2 dt \tag{6.46}$$

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we see that the above moment problems, equivalent to the control problem, can be solved with (6.46) finite, provided that

$$\sum_{k=-\infty}^{\infty} N_k^2 < \infty \, ,$$

which is the same as saying that the norm of the final state in $\hat{H}$ should be finite. We have, then, the exact controllability result that any $\hat{H}$ state may be controlled to any other $\hat{H}$ state during a time interval of length $T > 4$ with the control configuration we have described here. As discussed in connection with the wave equation in [FF], [GG], one cannot be sure that the state of the system remains in $\hat{H}$ for all $t \in [0,T]$. However, in the present case of the Maxwell equations one can show that these states do lie in

$H = H_{E,d}(R)$.

7. CONCLUDING REMARKS

The approximate controllability results of Section 5 would appear to be extendable to domains other than rectangular ones but the precise method of extension remains to be worked out. We will indicate some aspects of this problem which are clear from our current work.

First of all, the result of Section 5 is almost trivially extended to the case where control is exercised only on a subset $\{(\pi,y)|0 \leqslant a \leqslant y \leqslant b \leqslant \pi\}$, $b > a$, of $\{(\pi,y)|0 \leqslant y \leqslant \pi\}$. The only change is that the interval $1 + \delta \leqslant t \leqslant T - 1 - \delta$ appearing in (5.14) and subsequently must be modified to $d + \delta \leqslant t \leqslant T - d - \delta$ where

$$d = \inf_{\substack{a \leqslant y \leqslant b}} \left\{ \sup_{\substack{0 \leqslant \xi \leqslant \pi \\ 0 \leqslant \eta \leqslant \pi}} \left\{ [(\pi - \xi)^2 + (\eta - y)^2]^{1/2} \right\} \right\} .$$

If $\phi(\pi,y,t) \equiv \frac{\partial \phi}{\partial x}(\pi,y,t) \equiv 0$ for $\delta \leqslant t \leqslant T - \delta$, $a \leqslant y \leqslant b$, the Holmgren theorem will still apply to show that $\phi(x,y,t) \equiv 0$, $(x,y) \in R$, $d + \delta \leqslant t \leqslant T - d - \delta$. After that the remainder of the proof is the same: the same eigenfunctions and frequencies must be dealt with, the functions $\sin jy$, $\cos jy$ are still independent on $a \leqslant y \leqslant b$ if $b > a$ and the conditions

$$\hat{v}_0(x,y) = 0, \quad (x,y) \in B - \{(\pi,y)|a \leqslant y \leqslant b\}$$

$$\frac{\partial \hat{v}_0}{\partial x}(\pi,y) = 0, \quad a \leqslant y \leqslant b ,$$

still show $\hat{v}_0(x,y) \equiv 0$ in R.

The first limitation of the method which we have used in Section 5 lies in its dependence on the construction of $\phi(x,y,t)$ as a linear combination of partial derivatives of $\hat{v}$ and $\hat{w}$. It is necessary to have <u>a solution of the wave equation to which Holmgren's theorem may be applied</u>. This part of the proof can still be used for non-rectangular domains as long as a portion of the boundary on which control is applied is a straight line segment. Assuming the segment parallel to the y-axis, one can construct ϕ by the formula (5.8) again and show that ϕ and $\frac{\partial \phi}{\partial x}$ both vanish on the straight line segment in question, allowing subsequent application of the Holmgren theorem to show $\phi(x,y,t) \equiv 0$

for $(x,y) \in R$ and t in some interval $d + \delta < t < T - d - \delta$, with d depending on the geometry of R. But then we are faced with a second limitation.

The second limitation of the method which we have used lies in its reliance on the specific form of the eigenfunctions and frequencies to pass from $\phi(x,y,t) \equiv 0$ to the conclusion that both $\hat{v}(x,y,t)$ and $\hat{w}(x,y,t)$ are likewise identically zero. It needs to be emphasized that no local analysis will suffice here. In the one dimensional case (see our remarks at the end of Section 3) if the control problem is stated for boundary conditions

$$v(0,t) = 0, \quad \frac{\partial v}{\partial t}(1,t) = \alpha u(t) \tag{7.1}$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = \beta u(t) \tag{7.2}$$

the $\tilde{v}, \tilde{w}$ constructed as in Section 4 will satisfy the wave equation and

$$\tilde{v}(0,t) = 0, \quad \frac{\partial \tilde{v}}{\partial t}(1,t) = 0 , \tag{7.3}$$

$$\frac{\partial \tilde{w}}{\partial x}(0,t) = 0, \quad \frac{\partial \tilde{w}}{\partial x}(1,t) = 0 , \tag{7.4}$$

$$\alpha \frac{\partial \tilde{v}}{\partial x}(1,t) + \beta \frac{\partial \tilde{w}}{\partial t}(1,t) \equiv \phi(1,t) = 0 \tag{7.5}$$

Here if we take $\tilde{w}$ to be a non-zero solution of the wave equation satisfying (7.4) and take

$$\tilde{v}(x,t) = -\frac{\beta}{\alpha} \int_0^x \frac{\partial \tilde{w}}{\partial t}(\xi,t)d\xi$$

we clearly have $\tilde{v}(0,t) = 0$,

$$\frac{\partial \tilde{v}}{\partial t}(1,t) = \frac{\beta}{\alpha} \int_0^1 \frac{\partial^2 \tilde{w}}{\partial t^2}(\xi,t)d\xi$$

$$= -\frac{\beta}{\alpha} \int_0^1 \frac{\partial^2 \tilde{w}}{\partial \xi^2}(\xi,t)d\xi = \frac{\beta}{\alpha} \left(\frac{\partial \tilde{w}}{\partial x}(0,t) - \frac{\partial \tilde{w}}{\partial x}(1,t) \right) = 0 ,$$

$$\frac{\partial^2 \tilde{v}}{\partial t^2}(x,t) = -\frac{\beta}{\alpha} \int_0^x \frac{\partial^3 \tilde{w}}{\partial t^3}(\xi,t)d\xi$$

$$= -\frac{\beta}{\alpha} \int_0^x \frac{\partial^3 \tilde{w}}{\partial t \partial \xi^2}(\xi,t)d\xi = -\frac{\beta}{\alpha} \frac{\partial^2 \tilde{w}}{\partial t \partial x}(x,t) = \frac{\partial^2 \tilde{v}}{\partial x^2}(x,t)$$

so that $\tilde{v}$ satisfies the wave equation and, clearly, (7.5) is also satisfied. Thus the wave equation with (7.1), (7.2) is not approximately controllable; $\phi(x,t) \equiv \alpha \frac{\partial \tilde{v}}{\partial x}(x,t) + \beta \frac{\partial \tilde{w}}{\partial t}(x,t) \equiv 0$ but this does not imply that $\tilde{v}$ or $\tilde{w}$ are identically equal to zero. The additional condition which makes this work in (3.7) ff. is the fact that one can show there that

$$-\alpha \frac{\partial \tilde{v}}{\partial x}(0,t) + \beta \frac{\partial \tilde{w}}{\partial t}(0,t) = 0 .$$

It seems likely that the question of whether or not $\phi = 0$ implies that both $\hat{v}$ and $\hat{w}$, equivalently $\tilde{v}$ and $\tilde{w}$, are both zero must eventually reduce to a boundary value problem of an as yet unidentified type.

At the present writing there is only one, rather curious, result which we can offer which yields approximate controllability for a domain R of rather general shape. We suppose that the "control boundary" B_1, $B = \partial R$ includes two nonparallel line segments, ℓ_1 and ℓ_2, with unit exterior normals ν_1 and ν_2. Proceeding as before we can show, applying the Holmgren theorem together with

$$\frac{\partial \hat{v}}{\partial t} = 0 \quad \text{on} \quad \ell_1, \ell_2$$

$$\frac{\partial \hat{w}}{\partial \nu_i} = 0, \quad i = 1,2 \quad \text{on} \quad \ell_1, \ell_2, \quad \text{respectively,}$$

$$\alpha \frac{\partial \hat{v}}{\partial \nu_i} + \beta \frac{\partial \hat{w}}{\partial t} = 0, \quad i = 1,2 \quad \text{on} \quad \ell_1, \ell_2 \quad \text{respectively,}$$

that both

$$\phi_1 = \alpha \frac{\partial \hat{v}}{\partial \nu_1} + \beta \frac{\partial \hat{w}}{\partial t} , \tag{7.6}$$

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$$\phi_2 = \alpha \frac{\partial \hat{v}}{\partial \nu_2} + \beta \frac{\partial \hat{w}}{\partial t} \ , \tag{7.7}$$

must vanish identically in R for $d + \delta < t < T - d - \delta$, $\delta > 0$ arbitrary, $d > 0$ depending on the geometry of R and B, the location of ℓ_1 and ℓ_2 within B, etc. But then both ϕ_1 and ϕ_2 must vanish on ℓ_1 (say) for these values of t. Subtracting (7.6) from (7.7) we see that

$$\alpha \left(\frac{\partial \hat{v}}{\partial \nu_1} - \frac{\partial \hat{v}}{\partial \nu_2} \right) = 0 \quad \text{on} \quad \ell_1 \times [d + \delta, T - d - \delta]$$

This shows, since ℓ_1 and ℓ_2 are not parallel, that a nontangential derivative of $\hat{v}$ vanishes on $\ell_1 \times [d + \delta, T - d - \delta]$. Combining this with $\frac{\partial \hat{v}}{\partial t} = 0$ on ℓ_1 and applying the Holmgren theorem to $\hat{v}$ alone, much as in [5], [13], we are able to conclude $\hat{v} \equiv 0$, provided T is appropriately large. Then one easily has the same result for $\hat{w}$ and approximate controllability follows.

This result gives approximate controllability for R equal to the interior of any closed polyhedron in R^2 with control on at least two sides.

Further inspection of this argument shows that only ℓ_2 needs to be assumed to be a line segment. That is needed in order to identify ϕ_2 as a solution of the wave equation. We may then take ℓ_1 to be any smooth portion of B_1 which is never parallel to ℓ_1 and achieve the same result.

Finally, let us indicate that we are very much aware of the limitations, from the point of view of actual implementation, of the control configuration discussed in this paper. In principle, at least, the boundary conditions (1.7), (1.8), along with the further "single layer" condition discussed in connection with Figure 3.1, could be achieved with conducting bars attached to terminals as shown in Figure 3.
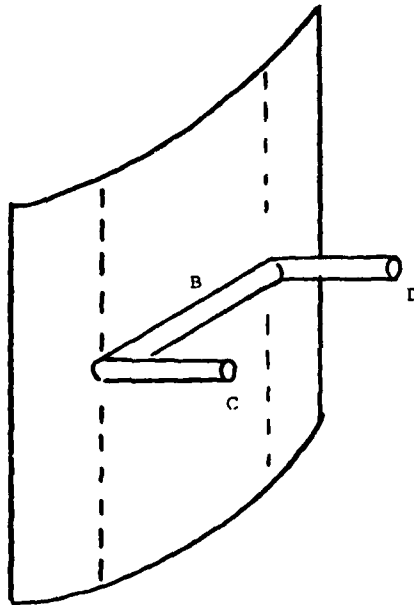
Figure 3. Conducting Bar and Busses

The perfectly conducting busses perpendicular to the boundary of Ω ensure that the normal component of $\hat{E}$, E_ν, is zero just outside Ω, provided that no net change is allowed to accumulate at the boundary of Ω, i.e., in the conducting bar. Thus the potentials at C and D must be regulated so that the potential difference C - D ensures the correct controlling current through the surface bar B while C + D is set so that there is no accumulation of charge at the bounding surface.

We have not considered any effects of propagation delays in the controlling circuits - i.e., we have not assumed that these are distributed parameter systems. This assumption, and evident limitations on the speed with which prescribed currents can be computed and established in the controlling circuits together with sensing limitations, place admittedly severe limitations on what can be done "open loop". It is likely that the eventual significance of our results will be most evident in connection with closed loop behavior wherein time varying magnetic fields $\hat{H}$ near the boundary of Ω induce currents in the bars B which, being resistive, will then act as energy dissipators. We hope to discuss this topic in later work.

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Another control configuration may be obtained by supposing the boundary of Ω to be a perfectly conducting sheet of material to which electromagnets are attached in a dense array as shown in Figure 4.
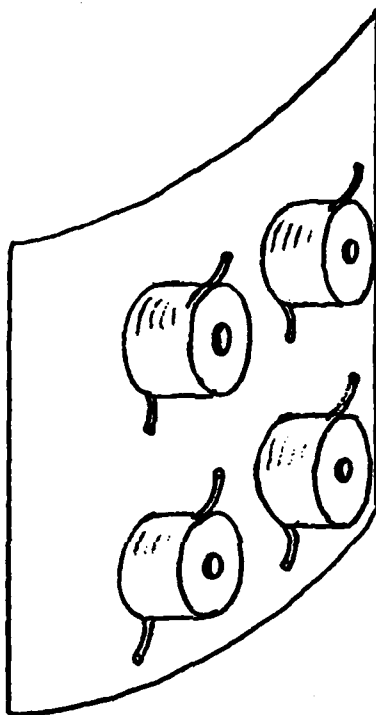

Figure 4. Electromagnet Array

If J denotes the current through the windings of the electromagnets, then we shall have

$$\vec{E}_\tau = 0$$

and

$$H_\nu = \alpha J$$

where α is dependent on the electromagnet's configuration. The theory in this case will take much the same form as the one discussed in this paper.

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     In a cylindrical region we consider electromagnetic fields independent of the axial coordinate: controlling the time evolution of such fields by means of boundary currents, likewise independent of the axial direction, is equivalent to controlling, simultaneously, two wave equations; one with boundary control of Dirichlet type, the other of Neumann type. In this paper we provide a preliminary study of control problems of this type and indicate what is necessary for extensions of our work.

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